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Contents

Abstract

This paper derives a general framework for collateral risk control determination in repurchase transactions or repos. The objective is to treat consistently heterogeneous collateral so that the collateral taker has a similar risk exposure whatever the collateral pledged. The framework measures the level of risk with the probability of incurring a loss higher than a pre-specified level given two well known parameters used to manage the intrinsic risk of collateral: marking to market and haircuts. It allows for the analysis in a self contained closed form of the way in which different relevant factors interact in the risk control of collateral (e.g. marking to market frequency, level of volatility of interest rates, time to capture and liquidity risk, probability of default of counterparty, etc.). The framework, which combines the recent theoretical literature on credit and interest risk, provides an alternative quantifiable and objective approach to the existing more ad-hoc rule-based methods used in haircut determination.

JEL Classification: E50, E58, G21, G10

Keywords: Collateral, Repurchase Transactions, Default Risk, Central Banks, Monetary Policy Operations

Non Technical Summary

This paper builds a general framework for risk control determination of collateral used in repurchase agreements or repos. The goal in building this framework is that heterogenous collateral should be treated in a consistent manner from a risk management point of view so that the collateral taker has a similar risk exposure and similar economic impact whatever the collateral pledged. In other words, the level of risk taken by the collateral taker should be homogenous across different types of collateral (e.g. bonds or equities).

The framework measures the level of risk with the probability of incurring a loss higher than a pre-specified level given two well-known parameters used to manage the intrinsic risk of collateral. These two parameters commonly used in the financial industry are *marking to market* (which helps reduce the level of loss by revaluing more or less frequently the collateral using market prices) and *haircuts* (which help reduce the level of loss by reducing the collateral value by a certain percentage). Given specified values for marking to market and haircut the framework presented is able to provide a probability of incurring a loss higher than allowed. Similarly, if the collateral taker fixes a level for the probability of incurring a loss higher than a prespecified level (i.e. its risk appetite) and a marking to market policy, the framework is able to provide a haircut level consistent with the probability of loss specified.

The framework developed in this paper builds on recent results in the financial economics literature in the area of credit and interest risk modelling. We rely on the so-called reduced form approach to model credit risk in which the counterparty entering the repurchase transaction defaults in an exogenous manner. In other words, the counterparty might default due to a variety of reasons that are independent from the specific repurchase transaction considered. The collateral asset value is assumed to change following the evolution of an interest rate model in the case of fixed income assets and a geometric brownian motion in the case of equity. This structure allows, compared to other methods presented in the literature, the derivation of a closed form solution for the measurement of the probability of incurring a loss higher than allowed given a marking to market and a haircut policy.

The framework allows for the analysis in a self-contained closed form solution of the way in which different relevant factors interact in the risk control of collateral and it confirms many practical intuitions. For example, it supports the perception that a higher haircut level should be required to cover for riskier collateral and that market to market frequency and haircuts are substitutes. In addition, the higher the probability of default of the counterparty and the level of volatility of interest rates affect directly the haircut. The time to capture or the time span needed before actual liquidation of the assets also affects positively the level of risk. Liquidity risk or the risk of incurring a loss in the liquidation due to illiquidity of the assets is also studied. A model for endogenous liquidity risk or the liquidity risk induced by one's owns action (e.g. through the sell of a large position) is introduced in the framework. Although the modeling of liquidity risk is far from settled in the literature, the framework is able to support the intuition that higher liquidity risk would aggravate the probability of loss. In this sense, the framework could be used as an objective basis to understand and quantify the relationship between a number of factors and the level of haircuts which is important not only for practitioners and academics but also for regulators.

Current industry practice still relies on ad-hoc rule based methods to establish the level of risk controls such as haircuts. On the whole, despite recent advances in the financial modeling of risks and incipient research in the area of risk control of collateral, the discussion on the precise level of risk mitigation achieved by accepting collateral is still rather vague. In addition, haircut levels used by market agents have been generally set without a consistent approach. We hope that this paper would stir further interest in an area that has been subject to still little research but whose importance is key in today's financial markets.

1 Introduction

A very important type of credit transaction in financial markets is the so-called repurchase transaction or repo. In this type of operation, an agent buys or sells eligible assets (i.e. collateral) against cash under repurchase agreements. According to the 2002 ISMA European repo market survey, the estimated total value of the outstanding repo business in Europe is equivalent to EUR 3.3 trillion.

The so-called collateral leg of the transaction plays a crucial role in this type of operations. The collateral leg is intended to mitigate the credit risk or default risk of the counterparty borrowing the cash. In case of default of the counterparty, the collateral taker can make good any loss by selling the collateral received. When the collateral considered is default-risk free (but not market-risk free, as with government bonds), collateralization is a way to transform credit risk (or risk of default) into market risk (more studied in the literature and easier to handle) as the default risky counterparty provides a default-risk free asset to guarantee its position. Even when the collateral itself can be defaultable (as in corporate bonds, OTC derivatives or loans), the credit risk is strongly mitigated.

The goal of this paper is to establish a general framework that would help the treatment of heterogenous collateral in a consistent manner from a risk management point of view in the context of repurchase transactions. Common practices have appeared to manage the intrinsic risk of collateral: marking to market[1] (which helps reduce the level of loss by revaluing more or less frequently the collateral using market prices) and haircuts[2] (which help reduce the risk of loss in case the counterparty defaults). In this paper, we consider both techniques in the establishment of the risk control determination framework with the aim of guaranteeing to the collateral taker a similar risk exposure and similar economic impact whatever the type of collateral pledged.

We use the vast theoretical literature on credit risk and on interest rate risk to provide a solution to our problem of defining a consistent framework to collateral risk control policies. Two approaches to modelling credit risk currently dominate financial economics: structural form and reduced-form approaches. The structural form approach extends the Merton (1974) option-based framework and models credit risk as a short position in a put option (and in practice this relies on the microeconomics of the firm's solvency situation). Reduced-form approaches, as settled by Madan and Unal (98), Jarrow et al. (97) and

[1] Marking to market refers to the practice of a periodical monitoring of the value of the collateral . If the difference between the collateral and the value of the notional amount (value of the loan) is smaller than a determined trigger level, the counterparty will be required to add some additional collateral. If the opposite happens, the amount of collateral can be decreased.

[2] Increases in collateral required depend on collateral type. For example, equities may be valued at 60% of their market value for collateral purposes, hence implying a haircut on their value of 40%. Haircuts would typically be smallest for short maturity government bonds, increase for longer maturity bonds, and be larger for corporate bonds and equities. The relationship that these haircuts should have from one to the other is what this paper investigates.

Duffie and Singleton (99) consider an exogenous stochastic default, modelled traditionally by a jump process with stochastic intensity. Default is defined as the first jump of the default process (see Cossin and Pirotte (2000) for an overview of the credit risk literature and Cossin and Hricko (2000) for application to collateral policies of commercial banks).

While the finance academic literature has focused on the issue of pricing credit and interest risk quite extensively, little research exists on the impact of collateral on credit risk exposures and collateral haircut determination models. Margrabe (1978) mentions the similarity between an exchange option and a margin account and provides the pricing for a very simple marking-to-market. Stulz and Johnson (1985) study the impact of collateralisation on the pricing of secured debt using contingent claim analysis. More recently, Jokivuolle and Peura (2000) present a model of collateral haircut determination for bank loans. Their model is geared to providing adequate loan-to-value ratios, which is similar to the concept of haircut, using an structural credit risk approach. In a related research, Cossin and Hricko (2000), present a methodology for haircut determination also using a structural approach but with the final objective of pricing a credit risk instrument backed with collateral. Notwithstanding this research, current industry practice still relies on ad-hoc rule-based methods to establish the level of haircuts (see Cossin and Pirotte 2001 for a typical haircut schedule of a major bank). As Jokivuolle and Peura (2000) put it, "on the whole, we find the discussion on collateral haircuts and loan to value ratios up to date rather vague in many respects". For example, the precise degree of risk mitigation pursued by taking collateral is typically not defined and haircut levels used by market participants have been generally set without a consistent approach. The aim of this paper is to participate in clarifying this existing theoretical gap by providing a general framework that would incorporate the main relevant factors affecting the determination of haircuts in a quantifiable and objective manner using recent general results in the financial economics literature in the area of credit and interest risk modelling.

In this paper the framework presented focuses on the level of risk taken by the collateral taker with the aim of making it homogenous across different collateral. Based on the collateral haircuts and predetermined loss level, we derive the probability of incurring a loss higher than allowed. Accordingly, if the collateral taker fixes its risk exposure, which is related to its risk appetite, the required haircut can be extracted. We rely on the reduced form literature as opposed to the approaches followed by Cossin and Hricko (2000) and Jokivuolle and Peura (2000) which are based on structural form methods. In our framework, default is determined exogenously while marking-to-market is also introduced and margin calls are derived endogenously by the evolution of the underlying used as collateral. This feature makes the problem easier to be generalised and to obtain a closed form solution for the case of single collateral when the interest rate is assumed to follow affine type processes. In addition to the marking-to-market policy, other realistic features are taken into account like time to capture[3] and liquidity effects. We also study the case where there exists correlation between interest rate and the probabilities of default of the counterparty. The framework confirms many practical intuitions, such as the direct impact on the haircut level required to cover for riskier collateral and supports the intuition that the frequency of marking to market and collateral are substitutes. In this sense, the framework could be used as an objective basis to understand the relationship between a number of factors and the level of haircuts which is important not only for practioners and academics but also for regulators.

We proceed as follows: In section 2, we describe the basic set up, where marking to market policy is specified as well as the basic problem. In part 3 the single collateral model is developed. Section 4 presents some extensions for realistic implementation: liquidity effects, time to capture, other types of collateral (equities, loans), stochastic default processes, positive trigger levels and alternative interest rate models. In section 5 we conclude.

2 The Set-Up

We start with a simple set-up. There is a repurchase transaction contract between the collateral taker and one counterparty. The counterparty borrows cash (the underlying) from the collateral taker. For simplicity of presentation, we assume that no interest is bearing on the cash borrowed. The extension to a constant interest rate bearing is straightforward.[4] The collateral taker asks the counterparty for α_t units of a bond $B(t,T)$ as collateral. We assume that the time to maturity of the collateral is greater than the expiration date of the contract between the collateral taker and the counterparty. For every collateral pledged, there exists a haircut h. We will assume a K periods contract, where margin calls can happen K times (for example, over a period of two weeks, there could be 14 daily margin calls, or 10 if they can happen only on business days). We treat the general case with constant time intervals between possible margin calls (no week ends) but provide in appendix the solution for non constant time intervals, thus accounting for periods in which collaterals can not be called (such as during week ends). In this set up we will consider 0 as a trigger level for the margin call (i.e., as soon as the value of the collateral diverges from the underlying value, there is a margin call in order to reestablish equivalency). In the extensions developed underneath, the trigger level will be set to a constant

[3] With time to capture we refer to the period between a default declaration and when the collateral can be sold by the central bank. Some adjustments are needed due to the existence of market risk in this period.

[4] In practice, the underlying may not be constant indeed, and it would bear the interest which is set at the beginning and distributed evenly. Denote the interest rate for the operation is R, so the value of the underlying at t will be

$$U_t = U_0 e^{Rt}$$

Replacing the constant underlying value with the above in all equations underneath would provide the result.

d.

At the end of period k $(k = 1, 2, ..., K)$ we face three situations:

1. The underlying U_0 is smaller than the level of the adjusted collateral (taking the haircut in consideration), i.e. $U_0 < \alpha_{k-1}B(t_k, T)(1 - h)$, where α_{k-1} is the amount of collateral at the beginning of period k. In this situation, the counterparty will receive an extra amount of collateral back such that $U_0 = \alpha_k B(t_k, T)(1 - h)$. The contract continues.

2. The underlying U_0 is larger than the level of the adjusted collateral, i.e. $U_0 > \alpha_{k-1}B(t_k, T)(1 - h)$. In this situation, a margin call happens and the counterparty will be required to deposit more collateral such that $U_0 = \alpha_k B(t_k, T)(1 - h)$. On the other hand, there exists the exogenous probability that the counterparty defaults in period k. If the counterparty does not default during period k, it will post the extra collateral and the contract continues.

3. In the case where the margin call happens and the counterparty is required to deposit more collateral, if the counterparty defaults in this period, it will not be able to fulfil the requirement, and the collateral taker will have a maximum loss of $(U_0 - \alpha_{k-1}B(t_k, T))$. The contract will stop and enter into a liquidation process.

The collateral taker is interested in the third situation when the loss is larger than a predetermined level $\overline{L} = lU_0$, i.e. $U_0 - \alpha_{k-1}B(t_k, T) > lU_0$. In other words, the collateral taker is interested in the joint event $X \cap Y$, where X is that, at the end of period k, the collateral's value is lower than $(1 - l)\,U_0$, and Y is that the counterparty defaults in period k. Event X mainly reflects market risk, and event Y the credit risk of the counterparty. The total probability of $X \cap Y$ happening in the life of the contract is a measure of the risk the collateral taker would take, which is determined by the collateral taker's risk appetite. Meanwhile, the total probability is related to the haircut attached to the collateral, so the collateral taker can adjust the haircut h to manage the risk.

In this set-up, we will first compute the haircut necessary for one single collateral, a default-free zero coupon bond $B(t, T)$, to protect for a given level of risk. Once the haircuts are calculated, the set-up could be extrapolated to take into account collateral portfolios consisting of n default-free zero coupon bonds. This paper, however, only deals with the single collateral model and leaves the collateral portfolio model for future research.

3 Single Collateral Model

To start with, we assume that the interest rate follows a Vasicek process as defined originally in Vasicek (1977). We examine the possibility of other processes truly leading interest rates later. The widely used Vasicek model has the advantage over some alternatives (such as Cox-Ingersoll-Ross (1985), thereafter

CIR) to be able to represent any possible term structure. Its main limitation consists in its possibly allowing for negative interest rates, something CIR does not. The Vasicek model is of course very extensively used in the theoretical literature (most of the credit risk literature with stochastic interest rates uses that model, see for example Shimko and alii (1993),Longstaff and Schwartz (1995)) as well as in practice. We have as in Vasicek (1977):

$$dr_t = a(b - r_t)dt + \sigma_r dW_t \tag{1}$$

where the interest rate follows a mean reverting process. a is the speed of mean reversion, b the long run or equilibrium value that the rate reverts to in the long run with a speed of a, σ_r is the instantaneous volatility and dW is the increment of a Wiener process,

Equivalently, the interest rate can be written:

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + e^{-at}\sigma_r \int_0^t e^{au} dW_u \tag{2}$$

Denote B_t the price of the bond with maturity T at time t. When the short rate r_t follows Vasicek(1977), B_t is written as

$$B_t = e^{m_t - n_t r_t} \tag{3}$$

where both m_t and n_t are deterministic functions of time t :

$$n_t = \frac{1 - e^{-a(T-t)}}{a} \tag{4}$$

$$m_t = \frac{(n_t - T + t)(a^2 b - \frac{\sigma_r^2}{2})}{a^2} - \frac{\sigma_r^2 n_t^2}{4a} \tag{5}$$

Therefore, expressing the bond in its logarithmic form and substituting the value of r_t we found previously, we obtain

$$\ln B_t = m_t - n_t[b + (r_0 - b)e^{-at} + e^{-at}\sigma_r \int_0^t e^{au} dW_u] \tag{6}$$

Therefore

$$x_t = \ln B_t \sim N(m_t - n_t[b + (r_0 - b)e^{-at}], n_t^2 e^{-2at}\sigma_r^2 \int_0^t e^{2au} du) \tag{7}$$

Simplifying we obtain:

$$x_t = \ln B_t \sim N(m_t - n_t[b + (r_0 - b)e^{-at}], \frac{n_t^2 \sigma_r^2}{2a}(1 - e^{-2at})) \tag{8}$$

The life of the contract is divided into K marking to market periods with interval τ. At the beginning of period $k(k = 1, 2, ..., K)$, the collateral is readjusted such that

$$U_0 = (1-h)\alpha_{k-1}B_{k-1} \tag{9}$$

$$\alpha_{k-1} = \frac{U_0}{(1-h)B_{k-1}} \tag{10}$$

where α_{k-1} is the quantity of bond collateral required at the end of period $k-1$. We use B_{k-1} to indicate the value at the beginning of period k (immediately after collateral readjustment at the end of period $k-1$) of a bond with maturity at T.

At the end of period k, before rebalancing, the collateral taker's loss is determined by the difference between the underlying and the collateral

$$loss_k = U_0 - \alpha_{k-1}B_k = U_0 - \frac{U_0}{1-h}\frac{B_k}{B_{k-1}} \tag{11}$$

In order to calculate the corresponding haircut to apply, we need to compute the probability that the loss exceeds a predetermined level $\overline{L}$. We also define the loss level as a function of the underlying, or $\overline{L} = lU_0$. $\overline{L}$ or equivalently l defines the level of risk the collateral taker is willing to take. The model will give an equivalency between the level of risk taken, the probability that the losses exceed that level of risk and the collateral haircut chosen, so that the bank can choose any one variable and obtain the two others. Hence, looking at two bonds, the collateral taker will be able to establish their respective haircut so that the risk exposure taken with either bond is the same. It will thus not favor one bond versus another in terms of risk exposure as it may do with adhoc haircut schedules.

To establish this framework, we calculate the probability that the actual loss exceeds the predetermined level $\overline{L}$.

We thus start from $\overline{L} \leqslant loss_k$ which can be rewritten:

$$\overline{L} \leqslant U_0 - \frac{U_0}{1-h}\frac{B_k}{B_{k-1}} \Longleftrightarrow \frac{B_k}{B_{k-1}} \leqslant (1-l)(1-h) \tag{12}$$

where we know, as shown above, that

$$\begin{aligned} B_k &= e^{m_k - n_k r_k} \\ B_{k-1} &= e^{m_{k-1} - n_{k-1} r_{k-1}} \end{aligned} \tag{13}$$

Therefore

$$\ln\left(\frac{B_k}{B_{k-1}}\right) = (m_k - m_{k-1}) - (n_k r_k - n_{k-1} r_{k-1}) \tag{14}$$

Or equivalently, expressing r_k in terms of r_{k-1}

$$\ln\left(\frac{B_k}{B_{k-1}}\right) = (m_k - m_{k-1}) - n_k b(1 - e^{-a\tau}) + \quad (15)$$
$$(-n_k e^{-a\tau} + n_{k-1}) r_{k-1} - n_k \sigma_r e^{-ak\tau} \int_{(k-1)\tau}^{k\tau} e^{au} dW_u$$

From equation (2), we know that r_{k-1} can be written as

$$r_{k-1} = b + e^{-a(k-1)\tau}(r_0 - b) + \sigma_r \left(\frac{1 - e^{-2a(k-1)\tau}}{2a}\right)^{\frac{1}{2}} z_1 \quad (16)$$

where

$$z_1 = \frac{\int_0^{(k-1)\tau} e^{au} dW_u}{\left(\int_0^{(k-1)\tau} e^{2au} du\right)^{\frac{1}{2}}} \sim N(0,1) \quad (17)$$

On the other hand, we also know that

$$-\int_{(k-1)\tau}^{k\tau} e^{au} dW_u = \left(\int_{(k-1)\tau}^{k\tau} e^{2au} du\right)^{\frac{1}{2}} z_2 \quad \text{where } z_2 \sim N(0,1) \quad (18)$$

Therefore substituting into $\ln\left(\frac{B_k}{B_{k-1}}\right)$ we obtain

$$x = \ln\left(\frac{B_k}{B_{k-1}}\right) = \mu_k + \sigma_{1k} z_1 + \sigma_{2k} z_2 \quad (19)$$

where

$$\mu_k = (m_k - m_{k-1}) + \frac{1 - e^{-a\tau}}{a}\left(b e^{-a(T-k\tau)} + e^{-a(k-1)\tau}(r_0 - b)\right)$$
$$\sigma_{1k} = \frac{1 - e^{-a\tau}}{a} \sigma_r \left(\frac{1 - e^{-2a(k-1)\tau}}{2a}\right)^{\frac{1}{2}} \quad (20)$$
$$\sigma_{2k} = n_k \sigma_r \left(\frac{1 - e^{-2a\tau}}{2a}\right)^{\frac{1}{2}}$$

Define a new random variable z:

$$z = \frac{\sigma_{1k}}{\sigma_k} z_1 + \frac{\sigma_{2k}}{\sigma_k} z_2 \quad (21)$$

where

$$\sigma_k = \sqrt{\sigma_{1k}^2 + \sigma_{2k}^2} \quad (22)$$

Since z_1 and z_2 are independent standard normal random variables, it can be shown that z is also a standard normal random variable. Hence

$$x \sim N(\mu_k, \sigma_k^2) \quad (23)$$

Then, the probability that $loss_k$ is larger than the predetermined value $\overline{L}$ is:

$$\begin{aligned} \Pr ob(loss_k &> \overline{L}) = P(0 \leq \frac{B_k}{B_{k-1}} \leq (1-l)(1-h)) \\ &= P(-\infty \leq \ln \frac{B_k}{B_{k-1}} \leq \ln((1-l)(1-h)) \\ &= N(\frac{\ln((1-l)(1-h)) - \mu_k}{\sigma_k}) \end{aligned} \tag{24}$$

If Q represents the annualised probability of default, then τQ would represent the probability of default in each period. As expressed earlier, this probability of default is exogenous and therefore assumed independent of a margin call or a loss level. This assumption (a typical reduced form model assumption) appears better than the alternative, which would make the probability of default dependent on the margin call. The later assumption (typical of structural form models) would imply that the credit operation contract and the value of the collateral alone would push the counterparty to default (and would assume that the position is large enough to force the counterparty to default). However, the position taken in this approach is that the cash taker (a company or individual) might default for a wider variety of reasons which might be totally unrelated to the specific contract considered (e.g. a company might experience liquidity problems and file for bankruptcy). At the beginning of period k, the probability that the counterparty has not defaulted is $(1-\tau Q)^{k-1}$. The probability of the counterparty defaulting and the collateral taker incurring a loss larger than $\overline{L}$ in period k (i.e. the probability of a joint event) will be

$$P(k) = N(\frac{\ln((1-l)(1-h)) - \mu_k}{\sigma_k})\tau Q \tag{25}$$

Taking into account that the counterparty can only default once, the probability of incurring a loss larger than $\overline{L}$ in period k becomes

$$(1-\tau Q)^{k-1} P(k) \tag{26}$$

Hence, the total probability of incurring a loss larger than $\overline{L}$ with K marking to market periods will be:

$$\begin{aligned} P(loss &> \overline{L}) = P(1) + (1-\tau Q)P(2) + ...(1-\tau Q)^{K-1}P(K) \\ &= \sum_{k=1}^{K} (1-\tau Q)^{k-1} P(k) \end{aligned} \tag{27}$$

If the collateral taker fixes $\overline{L}$ and $\Pr(loss > \overline{L})$,the required haircut can be recovered from the above formula.

For example, a collateral taker may decide, in a certain interest environment, to protect itself against a loss of more than 5% with counterparties that have an average probability of default of one percent. It can use this framework to obtain the haircut that corresponds to this risk appetite for a collateral of a government bond of maturity 3 years. It can also calculate how much an increase in haircut should go with a collateral of maturity 10 years in order to obtain the same risk exposure. It can thus propose a consistent haircut schedule so that whatever the maturity of the bond proposed as collateral, the risk exposure of the bank remains the same.

Proposition 1 *For l big enough, given a fixed collateral haircut, the longer the maturity of the bond used as collateral, the higher is the probability of incurring a loss larger than $\overline{L}$. To obtain the same probability of incurring a loss larger than $\overline{L}$, longer term bonds require higher haircuts.*

Proof. See Appendix ■

As we expect, the model shows that, other things being equal, increasing the haircut will decrease the probability of incurring a large loss, and that the longer the maturity of the bond used as collateral, the higher the haircut required in order to have the same probability of incurring the loss.

This result is well understood by practitioners. It is classical to have haircut schedules designed so that longer term bonds face higher haircuts than shorter term bonds. Interestingly, the methodology here gives precisely the extra amount of haircut required for extra maturity in a rational way. It also gives the sensitivity of the haircut level to other parameters, such as the interest rate environment (level and volatility of the interest rate), the frequency of the marking to markets (which reflect the control environment of the process) as well as the probability of default (which reflects the quality of the counterparties considered).

The following section gives simple numerical examples of these sensitivities.

3.0.1 Example

Consider the following benchmark[5] case for the values of the main parameters in our model

b	a	r_0	T	h	Q	σ_r
0.05	0.25	0.04	10	0.01	0.01	0.04

With these values we compute what is the total probability of a loss of 5% ($l = 5\%$) for the collateral taker in a contract of 10 years with a single bond pledged as a collateral. In other words, starting with a value of 100 for the underlying (cash provided to the counterparty), what is the probability of having a loss of 5, when the collateral pledged is a government bond of maturity 1 year and of initial value of 100/99% = 101.01. We study the case of daily,

[5] We take as a benchmark typical parameters reported in the literature. These values should be only interpreted as an example since they are not the result of a calibration procedure. Similar values can be found in Neftci (2000) for instance.

weekly and monthly possible marking to markets with possible margin calls. The results give low probabilities that will need to be modified (and increased) by the introduction of realistic features such as time to capture, liquidity effects, etc. These are introduced in the following sections. Nonetheless, the numbers are instructive to look at for a first understanding of what drives risk exposures with collateral in place. Classical market risk factors (the interest rate variables here), credit risk factors (the probability of default here, which is extended to a stochastic default process in a later extension) and elements of collateral policy such as the haircut schedule all drive probabilities of losses. We obtain the following probabilities

Time	Probability
Daily	3.26858×10^{-18}
Weekly	1.01347×10^{-5}
Monthly	6.1385×10^{-4}

With these results as a benchmark, we analyze what is the effect of changing the values of the different parameters of the models. We choose very extreme cases to make differences more obvious.

Changes in T, maturity of the bond held as collateral

T	Daily	Weekly	Monthly
1.5	0.	0.	1.33392×10^{-9}
20	7.14632×10^{-6}	2.41159×10^{-5}	7.9913×10^{-4}

As expected, the longer the time to maturity of the bond, the higher the probability of having losses. The usual practice of asking for higher haircuts for longer maturity bonds is thus justified. The mismatch between the maturity of the bond and the maturity of the loan agreed has a strong impact on the probability of losses.

Changes in h, haircut on bond considered for collateral

h	Daily	Weekly	Monthly
0.1	0.	2.59421×10^{-17}	6.16681×10^{-7}
0.001	2.75417×10^{-14}	5.13204×10^{-11}	9.25418×10^{-4}

Also as expected, when the haircut is high , the probability of having a loss decreases, while if the haircut is small, the probability increases. Haircuts are a good way to reduce risk exposure. We provide hereby the exact impact of the haircuts considered.

Changes in Q, probability of default of the counterparty

Q=prob of default	Daily	Weekly	Monthly
0.01	3.16435×10^{-17}	9.72023×10^{-5}	5.89537×10^{-3}
0.0001	3.27929×10^{-19}	1.01774×10^{-6}	6.16348×10^{-5}

The quality of the counterparty considered plays of course a large role on the level of the haircuts that should be asked for. Most of the time, a collateral taker will be required to establish a haircut schedule that is independent of the quality of the counterparty. One approach will then be to set a haircut schedule linked to the worst exposures, or to an average exposure. The methodology provides for the level of exposure depending on the counterparties considered.

We next consider the impact of the interest rate environment. The volatility of the interest rate process seems to have a particularly strong impact on the haircut levels necessary (via the probability of losses).

Changes in r_0, initial interest rate value

r_0	Daily	Weekly	Monthly
0.01	3.54892×10^{-18}	1.10399×10^{-5}	1.87388×10^{-3}
0.08	2.93061×10^{-18}	9.03382×10^{-6}	2.04854×10^{-5}

When the initial interest rate decreases, then the probability of losses increases. This can be explained by the fact that, if the initial interest rate is very low (lower than the long term mean), there is a high probability that it will increase. If the interest rate increases, the price of the bond will decrease, therefore, the probability of incurring losses is higher. As noted inmore details underneath though, the impact of the difference with the long term mean is not analyzed separately here while it has a distinct effect as can be seen in equation (25).

Changes in b, long term mean of interest rates

b	Daily	Weekly	Monthly
0.1	3.22775×10^{-18}	9.95571×10^{-6}	6.00103×10^{-4}
0.01	3.30184×10^{-18}	1.02807×10^{-5}	6.25082×10^{-4}

We can observe that when the long term mean of the interest rate increases, then the probability of losses decreases. Notice though that from the formula (25) above (as both interest rate level and long term mean affect only the drift μ_k and not the other components of the probability of losses), the long term means has an impact via its absolute level (but a rather minimal impact as it is multiplied by an exponential of low value, especially for long maturity bonds) and more importantly via the distance between the current interest rate and the long term mean (as this is multiplied by a higher value). The two distinct effects are not separated well in our sensitivity analysis as the interest rate is fixed while the long term mean is modified.

Changes in a, speed of mean reversion

a	Daily	Weekly	Monthly
0.1	9.36419×10^{-9}	3.48408×10^{-4}	1.87388×10^{-3}
0.5	0.	7.1690910^{-11}	2.04854×10^{-5}

When the speed of convergence of the interest rate to its mean increases, then the probability of losses decreases. It is interesting to see that the speed of mean reversion has an important effect on the overall value of the probabilities of losses. This is, with the volatility of the interest rate process, one of the main drivers, as far as the interest rate environment is concerned, of the bank risk exposure.

Changes in σ_r, volatility of the interest rate process

σ_r	Daily	Weekly	Monthly
0.015	0.	9.14667×10^{-19}	1.613×10^{-5}
0.05	5.0507×10^{-13}	6.845×10^{-5}	1.0111×10^{-3}

Changes in the volatility of the interest rate is one of the most important factors affecting the probability of loss. When the volatility increases, then the probability of losses also increases. In the benchmark we considered a quite high volatility ($\sigma_r = 0.04$) in order to see more easily how the probabilities change when we modify the different parameters.

While these numerical examples give us a first sense of what affects collateral policies, much needs to be incorporated to make it more realistic. First and foremost may be the fact that counterparties typically post several collaterals (portfolios of collaterals) as pledged. This issue is considered next.

4 Extensions for Realistic Implementation

4.1 Introduction of time to capture and liquidity risk

When the counterparty defaults, the collateral taker has to wait some time for the legal procedures to yield actual possession (and sale or liquidation) of the collateral. We call this phenomenon the time to capture. Suppose the time to capture is equal to δ marking to market intervals, i.e. $\delta\tau$, and the counterparty defaults during period k, so there is no rebalance at the end of period k. The loss of the collateral taker will not depend on the value of the collateral at the end of the $k\tau$ period, but on the value of collateral at the end of period $(k+\delta)\tau$.

On the other hand, one of the main concerns for the collateral taker will be, in the case of default, of how much cash can be obtained if the collateral position is liquidated. Unfortunately, this value is exposed to risk itself since it is very likely that the difference between the initial market value and the value realized after liquidation is greater than 0. This is what is called liquidity risk.

We can distinguish between *exogenous liquidity risk* (normally associated to the difference between bid and ask), which is due to the characteristics of the market and it is unaffected by the actions of any participant, and *endogenous liquidity risk* which refers to how individual actions can affect the price. Some collateral takers write in their books the bid price of the collateral, hence the exogenous liquidity risk disappears[6]. For those cases, only endogenous liquidity risk is taken into account. This risk consists in the possibility that the trades move the market in an adverse direction. In case of large positions, it is possible that the collateral taker sale would in itself affect the liquidity risk. The model of endogenous liquidity risk is still an open question in the literature and it is far from the scope of this paper to try and solve it. We will simply take the liquidity risk as a percentage loss when the collaterals are liquidated, i.e. liquidity loss is θ percent of the collateral value at the time of liquidation, which can be different for each collateral.

At the beginning of period k, $t = (k-1)\tau$, we have

$$U_0 = (1-h)\alpha_{k-1}B_{k-1} \tag{28}$$

then

$$\alpha_{k-1} = \frac{U_0}{(1-h)\,B_{k-1}} \tag{29}$$

At $t = (k+\delta)\,\tau$ the possible loss is equal to

$$Loss_k = U_0 - \alpha_{k-1}B_{k+\delta}(1-\theta) \tag{30}$$

Or equivalently, by substituting α_{k-1}

$$Loss_k = U_0 - \left[\frac{U_0}{1-h}\frac{B_{k+\delta}}{B_{k-1}}\right](1-\theta) \tag{31}$$

From the single collateral case, we know that

$$\ln\left(\frac{B_{k+\delta}}{B_{k-1}}\right) = \mu_{k,\delta} + \sigma_{k,\delta}z \tag{32}$$

where

$$z \ \tilde{}\ N(0,1)$$

$$\mu_{k,\delta} = (m_{k+\delta} - m_{k-1}) + \frac{1-e^{-a(\delta+1)\tau}}{a}(be^{-a(T-(k+\delta)\tau)} + e^{-a(k-1)\tau}(r_0 - b)) \tag{33}$$

$$\begin{aligned} \sigma_{k,\delta} &= \sqrt{\sigma_{1k,\delta}^2 + \sigma_{2k,\delta}^2} \\ \sigma_{1k,\delta} &= \frac{1-e^{-a(\delta+1)\tau}}{a}\sigma_r\left(\frac{1-e^{-2a(k-1)\tau}}{2a}\right)^{\frac{1}{2}} \\ \sigma_{2k,\delta} &= n_{k+\delta}\sigma_r\left(\frac{1-e^{-2a(\delta+1)\tau}}{2a}\right)^{\frac{1}{2}} \end{aligned} \tag{34}$$

Then, we have the following relationship between the predetermined maximum loss, $\overline{L} = lU_0$, and z.

[6] We have modeled in the appendix the case where exogenous liquidity risk does not disappear for the banks that do not follow the policy of keeping in their books the bid price of the collateral.

$$lU_0 \quad \leq \quad loss_k = U_0 - \left[\frac{U_0}{1-h} e^{\mu_{k,\delta}+\sigma_{k,\delta} z}\right](1-\theta) \tag{35}$$

$$\Longrightarrow \quad z \leq \frac{\ln\left(\frac{(1-l)(1-h)}{1-\theta}\right) - \mu_{k,\delta}}{\sigma_{k,\delta}} \tag{36}$$

The probability of incurring a loss larger than $\overline{L}$ will be

$$P(k) \quad = \quad P(loss_k \geq \overline{L})\tau Q \tag{37}$$

$$= \quad N\left(\frac{\ln\left(\frac{(1-l)(1-h)}{1-\theta}\right) - \mu_{k,\delta}}{\sigma_{k,\delta}}\right)\tau Q \tag{38}$$

Therefore, the probability of incurring a loss larger than $\overline{L}$ in the life of contract will be

$$P(loss \geq \overline{L}) = \sum_{k=1}^{K}(1-\tau Q)^{k-1}P(k) \tag{39}$$

It is not difficult to find that the similarity between this extension with the original case. Introduction of liquidity risk θ is simply to replace $(1-l)(1-h)$ with $\frac{(1-l)(1-h)}{1-\theta}$. In other words, the impact of liquidity risk θ is to adjust l or h to corresponding smaller numbers $\frac{l-\theta}{1-\theta}$ or $\frac{h-\theta}{1-\theta}$, consequently to increase the probability of incurring a loss larger than $\overline{L}$.

Introduction of time to capture δ is to replace μ_k and σ_k with $\mu_{k,\delta}$ and $\sigma_{k,\delta}$ accordingly. Since time to capture δ is always positive, it adds more uncertainty and increase the probability of incurring a loss larger than $\overline{L}$. When δ is large, the uncertainty during time to capture will be larger than the uncertainty during one marking to market interval, so that the risk due to the time to capture will be an important part (possibly dominant part) of the overall risk.

We thus have derived the first important extension of the model, integrating the realistic features of a time to capture and liquidity effect.

4.2 Examples

In this section, we analyze the effect of introducing time to capture and liquidity effects in a single collateral context.

Consider the following benchmark case for the values of the main parameters in our model

b	a	r_0	T	h	Q	σ	$loss$	Time to capture	liquidity
0.05	0.25	0.04	10	0.01	0.01	0.015	0.05	1 month	0.03

These values imply that the collateral will not be captured before a month after the event of default. This may be due to legal constraints that will make it impossible to liquidate the collateral before or special characteristic of the financial asset used as collateral which may require a long analysis period by the prospective buyers (e.g. asset backed securities with special covenants). This problem is combined with the liquidity issue, in which a loss of 3% of value of the bonds occurs at liquidation, in case of a thin market.

We first have a look to the effect of different marking to market intervals:

Marking to Market	Daily	Weekly	Monthly
	2.10434×10^{-3}	2.22007×10^{-3}	2.66116×10^{-3}

We can observe that with the introduction of the time to capture and the liquidity effect, the probabilities of loss have increased considerably. We further analyze the effects of changing the time to capture interval and the liquidity level under a daily marking to market set up.

Time to capture

Time to capture	Prob of loss
2 weeks	1.35211×10^{-3}
2 months	2.65833×10^{-3}

The probability of loss increases when we increase the time to capture interval. This is due to the fact that we face more uncertainty about the price of the collateral at the moment of the liquidation.

Liquidity

Liquidity	Prob of loss
0	1.25153×10^{-3}
0.02	9.91587×10^{-3}

As expected, the probability of loss has increased when we increase the liquidity loss level. Clearly, some instruments (such as loans) will face strong liquidity problems. Our methodology thus allows to differentiate between instruments for which liquidity is good and instruments for which liquidity is poor. The haircut can thus be set to compensate for that problem.

4.3 Further Extensions

4.3.1 Other instruments as collateral: Equity

In this section we introduce the use of equities as collateral.

As classical since Black and Scholes(1973), we assume the stock price S_t follows

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \qquad (40)$$

or equivalently

$$S_t = S_u e^{\int_u^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_u^t \sigma_s dW_s}, \qquad \forall\ u < t \tag{41}$$

where μ_t and σ_t are deterministic and bounded functions of time t. It can be shown that

$$\ln\left(\frac{S_t}{S_u}\right) \sim N\left(\int_u^t (\mu_s - \frac{1}{2}\sigma_s^2)ds, \int_u^t \sigma_s^2 ds\right) \tag{42}$$

Following the same methodology we have used until now, and assuming K marking to market periods with interval τ, at the end of period $k-1$ $(k = 1, 2, ..., K)$,

$$U_0 = (1-h)\alpha_{k-1}S_{k-1} \tag{43}$$

$$\alpha_{k-1} = \frac{U_0}{(1-h)S_{k-1}} \tag{44}$$

where as before α_{k-1} is the quantity of equity collateral required at the end of period $k-1$. At the end of period k, before rebalancing, the loss is determined as:

$$loss_k = U_0 - \alpha_{k-1}S_k = U_0 - \frac{U_0}{(1-h)}\frac{S_k}{S_{k-1}} \tag{45}$$

Therefore, applying the same methodology we used for the bond collateral case we obtain:

$$\begin{aligned} \Pr ob(loss_k &> \overline{L}) = P(0 \leq \frac{S_k}{S_{k-1}} \leq (1-l)(1-h)) \\ &= P(-\infty \leq \ln\frac{S_k}{S_{k-1}} \leq \ln((1-l)(1-h)) \\ &= N\left(\frac{\ln((1-l)(1-h)) - \int_{(k-1)\tau}^{k\tau}(\mu_s - \frac{1}{2}\sigma_s^2)ds}{\sqrt{\int_{(k-1)\tau}^{k\tau}\sigma_s^2 ds}}\right) \end{aligned} \tag{46}$$

The total probability of incurring a loss can be easily derived from the above formula using the methodology described in previous sections.

4.3.2 Poisson process for default of the counterparty

For simplicity, we have assumed for the time being a constant probability of default for the counterparty considered. This is a simplification of reality as default probabilities are not constant, but rather stochastic. We use the classical formalization of reduced form models (see Jarrow and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1999)). The structure of the problem remains the same as above. Difficulties arise when the default process is correlated to the interest rate.

Assume there exists a jump process $J(t)$ with intensity process $\lambda(t)$. Default occurs at the first jump time of $J(t)$. If default happens at τ, this implies by definition

$$\tau := \inf \{t \geq 0 \mid J(t) = 1\} \tag{47}$$

The probability of no default event occurring between time t and T is given by

$$P[J(T) - J(t) = 0] = e^{-\int_t^T \lambda(s)ds} \tag{48}$$

This implies that the distribution of the default time is given by

$$P[\tau \leq T] = 1 - e^{-\int_0^T \lambda(s)ds} \tag{49}$$

and its density

$$f(t) = \lambda(t)e^{-\int_0^t \lambda(s)ds} \tag{50}$$

Assume that the default intensity is determined as follows:

$$\lambda_t = \lambda_t^o + \phi r_t \tag{51}$$

where λ_t^o is a stochastic variable that follows a mean-reverting diffusion process:

$$d\lambda_t^o = a_\lambda(b_\lambda - \lambda_t^o)dt + \sigma_\lambda dW_t^\lambda \tag{52}$$

Recall that default-free interest rate r_t follows

$$dr_t = a(b - r_t)dt + \sigma_r dW_t \tag{53}$$

Here W_t^λ and W_t are two independent Brownian motion, so $\{\lambda_t^o, 0 \leq t \leq T\}$ and $\{r_t, 0 \leq t \leq T\}$ are two independent stochastic processes. ϕ can be considered as a constant. Notice that the empirical literature tends to show a negative correlation between interest rate and intensity of default (see for example Duffee (98)). Given this model, we can compute as done in the other set-ups the probability of having a loss larger than lU_0 in period k. The calculations are provided in Appendix.

4.3.3 Introduction of non-zero trigger level

In practice, margin calls are not effected as soon as there is a difference in value between the underlying and the adjusted collateral (after haircut). A certain level of difference is allowed (1% in the case of the EuroSystem) to avoid having too frequent calls.

When the trigger level is zero, the collateral portfolio is revalued and adjusted at each marking-to-market to keep the same weights of its components.

Meanwhile, the new value of the collateral portfolio, adjusted after haircuts, is equal to that of the underlying. In the case of non-zero trigger levels, the collateral portfolio is revalued but not always adjusted at each marking to market period. When the value of the collateral portfolio, adjusted after haircuts, is inside a narrow range centered in the value of the underlying, no adjustment is required, and the same collateral portfolio will be kept to the next margin call. Otherwise, the collateral will be adjusted to keep its weights of components and its value, adjusted after haircuts, equal to that of the underlying.

It is very difficult to model the non-zero trigger level precisely because it strongly depends on the specific trajectory. We study the approximation represented by two boundaries, first in the case of a single collateral and then in the case of a collateral portfolio. Denote d the trigger level. At the beginning of period k,

$$(1-d)\,U_0 \leq (1-h)\,\alpha_{k-1}B_{k-1} \leq (1+d)\,U_0$$

So,

$$\frac{(1-d)\,U_0}{(1-h)\,B_{k-1}} \leq \alpha_{k-1} \leq \frac{(1+d)\,U_0}{(1-h)\,B_{k-1}}$$

At the end of period k,

$$U_0 - \frac{(1+d)\,U_0}{1-h}\frac{B_k}{B_{k-1}} \leq Loss_k = U_0 - \alpha_{k-1}B_k \leq U_0 - \frac{(1-d)\,U_0}{1-h}\frac{B_k}{B_{k-1}}$$

Equivalently,

$$\frac{(1-l)\,(1-h)}{1+d} \leq \frac{B_k}{B_{k-1}} \leq \frac{(1-l)\,(1-h)}{1-d}$$

Following the similar process, we will have

$$P(loss > \overline{L}) = \sum_{k=1}^{K}(1-\tau Q)^{k-1}P(k)$$

with

$$N(\frac{\ln\left[\frac{(1-l)(1-h)}{1+d}\right]-\mu_k}{\sigma_k})\tau Q \leq P(k) \leq N(\frac{\ln\left[\frac{(1-l)(1-h)}{1-d}\right]-\mu_k}{\sigma_k})\tau Q$$

We thus obtain an approximation of haircuts required (or risk exposures) depending on the trigger level considered.

4.4 Alternative interest rate model

In order to check the assumption of interest rate following a Vasicek type of process, we decide to try other interest rate model specifications. Concretely, we analyze the differences if we assume a CIR process for interest rate with our model. Unfortunately, we are not able to derive closed form solutions for the case of CIR, therefore, simulation methods are used. While Monte Carlo simulations allow for more flexibility in the modelling (with the possibility of using other interest rate models and more complex features), they do not allow for the use and speed of calculations of the analytical formulas derived above, nor do they allow for general results analysis and sensitivities as those provided above.

CIR assumes the following process for the interest rate:

$$dr_t = a'(b' - r_t)dt + \sigma'_r\sqrt{r_t}dW_t \tag{54}$$

At time t, the price of the bond can be found as:

$$B_t(T) = A_t(T)e^{-n'_t(T)r_t} \tag{55}$$

where

$$n'_t(T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a')(e^{\gamma(T-t)} - 1) + 2\gamma} \tag{56}$$

$$A_t(T) = \left[\frac{2\gamma e^{(a'+\gamma)(T-t)}/2}{(\gamma + a')(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2a'b'/\sigma'^2_r} \tag{57}$$

with

$$\gamma = \sqrt{a'^2 + 2\sigma'^2_r} \tag{58}$$

In order to compare the results of Vasicek model with the CIR model, we discretize the process followed by the interest rate and we simulate the randomness introduced by the Brownian Motion.

To check the error we are making with simulations, we also simulate the Vasicek process and compare it with the real values as obtained from the analytical results given above. We obtain an acceptable accuracy[7] (around 2% relative difference between theoretical Vasicek and simulated Vasicek results) with a time step of five times per day and a number of repetitions equal to 1000. We use these parameters in order to perform our CIR simulations.

However, before carrying out our study, we must consider a case where the parameters of Vasicek and CIR are compatible. For that we use the parameters specified in Chan et.al. (1992) where different interest rate models are estimated

[7] We measure the accuracy as the difference between the real value and the simulated value. We repeat the simulation 10 times. In each simulation the time step is of five times per day and the number of repetitions equals to 1000. Taking the average of the errors in every set of simulations we found that this is around 2%.

for the same period (from June 1964 to December 1989). The parameters found in the estimation (and the ones we use in our simulations) are:

	Vasicek	CIR
a	0.1779	0.2339
b	0.0867	0.080803
σ	0.02	0.08544

(59)

Based on this parameters, we study the differences between Vasicek and CIR models for different values of the haircut level, the level of loss accepted and the initial interest rate. The results follow in the table bellow:

- $r_0 = 0.08$

Vasicek case

	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$l = 0.001$	1.27159×10^{-5}	2.53348×10^{-4}	3.18439×10^{-4}	3.74778×10^{-4}
$l = 0.01$	4.95089×10^{-6}	1.56547×10^{-4}	2.07011×10^{-4}	2.53348×10^{-4}
$l = 0.05$	1.6815×10^{-6}	4.95089×10^{-6}	8.48165×10^{-6}	1.27159×10^{-6}
$l = 0.1$	2.87072×10^{-13}	1.79085×10^{-9}	4.43444×10^{-9}	8.90452×10^{-9}

CIR case

	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$l = 0.001$	1.74088×10^{-5}	2.34474×10^{-4}	2.93948×10^{-4}	3.56502×10^{-4}
$l = 0.01$	1.04118×10^{-5}	1.45685×10^{-4}	1.88916×10^{-4}	2.29979×10^{-4}
$l = 0.05$	4.16424×10^{-7}	1.0995×10^{-5}	1.49098×10^{-5}	1.84918×10^{-5}
$l = 0.1$	0	0	8.32847×10^{-8}	3.33194×10^{-7}

The numbers indicate the probabilities values under Vasicek model and CIR model.

We observe that for the standard case ($l = 0.01$ and $h = 0.01$) the differences between both models are rather small. We can then conclude that the utilization of the extensively used Vasicek model is not a drastic restriction although interest rate modeling may matter in some cases.

5 Conclusions

We have derived how a collateral taker should build collateral policies (and notably haircut schedules) for the collateral policy to be consistent. We thus provide for the framework a collateral taker should follow in order to treat different collaterals consistently. The exercise reveals itself to be quite complex when one aims at taking into account in the modeling the muldimensional complexity of real use: the presence of time to capture, of liquidity effects, of different classes of collateral, etc. In deriving the structure of consistent collateral policies, we also have obtained a number of results. The longer the maturity of the instrument considered, the higher the haircuts that should be effected. Marking to market intervals, interest rate dynamics, credit risk of the counterparty all interact in the determination of haircut schedules. The framework could be used as an objective basis to understand the relationship between these factors (liquidity effects, credit risk of counterparty, time to capture, trigger levels, stochastic default processes, alternative interest rate models) and the level of haircuts. We hope that the framework presented here could trigger renovated interest in an area that has been subject of little research to date but whose importance is key in today's financial markets.

While we have derived how a consistent collateral policy should be constructed, we do not provide for what is the level of collateral policy that should be chosen. This will be decided by strategic and political factors as well as agency and other issues that can only be treated in a general equilibrium framework to provide full answers. In addition, the proposed framework for single collateral could be extrapolated to portfolio of collateral in future research to analyse the possible risk reduction gains that could be obtained and determine the corresponding haircut levels. Much research thus remains to be done to obtain what would be the optimal collateral policy for a collateral taker. Ideally, future research would combine the framework of collateral risk control determination as obtained here with optimality issues linked to auctioning problems and game theoretical agency and asymmetry of information issues to obtain a rational collateral policy. To achieve this goal, researchers will have to combine game theory with continuous time finance, a challenge that some researchers are currently aiming for.

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6 Appendices:

6.1 Appendix 1: Proof of proposition 1

We will proof in this section that bearing same risk, a collateral formed with a long term maturity bond requires a higher haircut than a collateral formed with a short term maturity bond.

Let

$$P(loss_k \geq lU_0) = N(\frac{\ln((1-l)(1-h)) - \mu_k}{\sigma_k}) = f(T, h) \tag{60}$$

where

$$\begin{aligned}
\mu_k &= (m_k - m_{k-1}) + \frac{1 - e^{-a\tau}}{a}(be^{-a(T-k\tau)} + e^{-a(k-1)\tau}(r_0 - b)) \\
\sigma_k &= \sqrt{\sigma_{1k}^2 + \sigma_{2k}^2} \\
\sigma_{1k} &= \frac{1 - e^{-a\tau}}{a}\sigma_r(\frac{1 - e^{-2a(k-1)\tau}}{2a})^{\frac{1}{2}} \\
\sigma_{2k} &= n_k\sigma_r(\frac{1 - e^{-2a\tau}}{2a})^{\frac{1}{2}}
\end{aligned} \tag{61}$$

Taking partial derivatives of function f with respect to h and T we obtain:

$$\frac{\partial f}{\partial h} = n(\frac{\ln((1-l)(1-h)) - \mu_k}{\sigma_k})\frac{(-1)}{1-h}\frac{1}{\sigma_k} \tag{62}$$

$$\frac{\partial f}{\partial T} = n(\frac{\ln((1-l)(1-h)) - \mu_k}{\sigma_k})\frac{-\frac{\partial \mu_k}{\partial T}\sigma_k - (\ln((1-l)(1-h)) - \mu_k)\frac{\partial \sigma_k}{\partial T}}{\sigma_k^2} \tag{63}$$

where $n(.)$ is the standard normal density function while $N(.)$ is the standard normal cumulative density function. We can already observe that $\frac{\partial f}{\partial h} < 0$. Continuing with the computations

$$\frac{\partial \mu_k}{\partial T} = \frac{\partial m_k}{\partial T} - \frac{\partial m_{k-1}}{\partial T} + \frac{1 - e^{-a\tau}}{a}be^{-a(T-k\tau)}(-a) \tag{64}$$

Remembering the expressions for m_k and n_k

$$m_k = \frac{(n_k(T) - T + k\tau)(a^2 b - \frac{\sigma_r^2}{2})}{a^2} - \frac{\sigma_r^2 n_k^2}{4a}$$

$$n_k = \frac{1 - e^{-a(T-k\tau)}}{a}$$

Computing the partial derivatives of n_k and m_k we obtain:

$$\frac{\partial n_k}{\partial T} = \frac{1}{a}(-e^{-a(T-k\tau)})(-a) = e^{-a(T-k\tau)} \tag{65}$$

$$\begin{aligned} \frac{\partial m_k}{\partial T} &= \frac{(a^2b - \frac{\sigma_r^2}{2})}{a^2}(e^{-a(T-k\tau)} - 1) - \frac{\sigma_r^2}{4a}2n_k e^{-a(T-k\tau)} \\ &= -abn_k + \frac{\sigma_r^2}{2}n_k^2 \end{aligned} \tag{66}$$

Therefore,

$$\begin{aligned} \frac{\partial \mu_k}{\partial T} &= -abn_k + \frac{\sigma_r^2}{2}n_k^2 + abn_{k-1} - \frac{\sigma_r^2}{2}n_{k-1}^2 \\ &\quad -b(1-e^{-a\tau})e^{-a(T-k\tau)} \end{aligned} \tag{67}$$

Note that

$$\begin{aligned} n_k(T) - n_{k-1}(T) &= \frac{-e^{-a(T-k\tau)} + e^{-a(T-k\tau+\tau)}}{a} \\ &= \frac{-e^{-a(T-k\tau)}}{a}(1-e^{-a\tau}) \end{aligned} \tag{68}$$

$$\begin{aligned} n_k^2 - n_{k-1}^2 &= (n_k + n_{k-1})(n_k - n_{k-1}) \\ &= \frac{-e^{-a(T-k\tau)}}{a}(1-e^{-a\tau})\frac{2 - e^{-a(T-k\tau)} - e^{-a(T-k\tau+\tau)}}{a} \end{aligned} \tag{69}$$

Hence,

$$\begin{aligned} \frac{\partial \mu_k}{\partial T} &= be^{-a(T-k\tau)}(1-e^{-a\tau}) + \frac{\sigma_r^2}{2}(n_k^2 - n_{k-1}^2) \\ &\quad -b(1-e^{-a\tau})e^{-a(T-k\tau)} \\ &= \frac{\sigma_r^2}{2}(n_k^2 - n_{k-1}^2) \end{aligned} \tag{70}$$

$$\begin{aligned} \frac{\partial \sigma_k}{\partial T} &= \frac{\sigma_{2k}}{\sqrt{\sigma_{1k}^2 + \sigma_{2k}^2}}\frac{\partial n_k}{\partial T}\sigma_r(\frac{1-e^{-2a\tau}}{2a})^{\frac{1}{2}} \\ &= \frac{\sigma_{2k}}{\sigma_k}e^{-a(T-k\tau)}\sigma_r(\frac{1-e^{-2a\tau}}{2a})^{\frac{1}{2}} \end{aligned} \tag{71}$$

Notice that

$$\begin{aligned} \frac{\partial \sigma_k}{\partial T} &> 0 \qquad (72) \\ \frac{\partial \mu_k}{\partial T} &< 0 \end{aligned}$$

Define loss lU_0 such that $P(loss_k \geq lU_0)$ is small (say, smaller than 5%) such that $\frac{\ln((1-l)(1-h))-\mu_k}{\sigma_k} < 0$. Then, we have that

$$\begin{aligned} \frac{\partial f}{\partial T} &= n(\frac{\ln((1-l)(1-h))-\mu_k}{\sigma_k})\frac{1}{\sigma_k}\left[-(\frac{\partial \mu_k}{\partial T}+\frac{(\ln((1-l)(1-h))-\mu_k}{\sigma_k})\frac{\partial \sigma_k}{\partial T})\right] \\ \frac{\partial f}{\partial T} &> 0 \qquad (73) \end{aligned}$$

We can express the total variation in the probability of having a loss as:

$$dP = \frac{\partial f}{\partial T}dT + \frac{\partial f}{\partial h}dh \qquad (74)$$

For the same probability (i.e., total variation is equal to 0), we have

$$\begin{aligned} dP &= \frac{\partial f}{\partial T}dT + \frac{\partial f}{\partial h}dh = 0 \Longrightarrow \qquad (75) \\ \frac{dh}{dT} &= -\frac{\frac{\partial f}{\partial T}}{\frac{\partial f}{\partial h}} \end{aligned}$$

Since we have shown that $\frac{\partial f}{\partial T} > 0$ and $\frac{\partial f}{\partial h} < 0$, we have $\frac{dh}{dT} > 0$ which implies that $h_2 > h_1 \Leftrightarrow T_2 > T_1$ for a fixed level of probability of loss, i.e. for the same level of probability of loss, the bond with longer time to maturity requires a higher haircut than the bond with shorter time to maturity.(Q.E.D.)

6.2 Appendix 2: Exogenous liquidity risk

Concerning the liquidity risk, we can distinguish between *endogenous liquidity risk*, which refers how individual actions can affect the price, and *exogenous liquidity risk* which is due to the characteristics of the market and it is unaffected by the actions of any participant.

In the paper we have developed the concept of endogenous liquidity risk. This risk consists in the possibility that the trades move the market in an adverse direction. We consider this risk when the size of the position overcomes the quote depth, where quote depth is defined as the volume of the asset available at the market maker's quoted price (bid or ask). In very liquid assets, the level of quote depth is very high which implies that the majority of the transactions will not pass this level and hence will not face this risk. In a very simple approach, we have modeled this risk as a constant that decreases the value of the bond.

On the other hand, exogenous liquidity risk is common to all the participants in the market. It is normally related to the difference between bid and ask. In very liquid securities, this difference will be very small, and therefore this risk is also small.

In this appendix we want to add the concept of exogenous liquidity risk since it is relevant for all kind of transactions, no matter their size. Moreover, the data needed to quantify this exogenous liquidity risk is normally available.

To introduce the exogenous liquidity risk, we will follow basically the methodology developed by Bangia et al. (98) adapting it to our model. The mentioned paper assumes that the cost of liquidity is based on a certain average spread plus a multiple of the spread volatility, to cover most of the spread situations. Concretely, the cost of exogenous liquidity is

$$\frac{1}{2}\left[B_t(\overline{S}+a\widetilde{\sigma})\right] \tag{76}$$

where B_t is today's mid-price for the bond pledge as collateral, $\overline{S}$ is the average relative spread (relative spread is defined as $\frac{Ask-Bid}{Mid}$), $\widetilde{\sigma}$ is the volatility of the relative spread and a is a scaling factor such that we achieve roughly the desired probability coverage. All the parameters taking part to determine the cost of exogenous liquidity are observable except a which depends on the distribution function of the relative spread. Hence, it would be no difficult to incorporate this risk into our model.

By equation (11) we knew that the loss was determined as

$$loss_k = U_0 - \alpha_{k-1}B_k\left(T\right)$$

With the introduction of liquidity risk, this possible loss must increase as follows:

$$loss_k = U_0 - \alpha_{k-1}B_k\left(T\right) + \alpha_{k-1}\frac{1}{2}\left[B_k\left(T\right)\left(\overline{S}+a\widetilde{\sigma}\right)\right] \tag{77}$$

We again have to check whether the $loss_k$ is greater or equal to the predetermined value $\overline{L}$, i.e.

$$\begin{aligned} \overline{L} &\leqslant U_0 - \frac{B_k\left(T\right)}{B_{k-1}(T)}\frac{U_0}{(1-h)}\left[1-\frac{1}{2}(\overline{S}+a\widetilde{\sigma})\right] \Longleftrightarrow \\ \frac{B_k\left(T\right)}{B_{k-1}\left(T\right)} &\leqslant (1-l)\frac{(1-h)}{\left[1-\frac{1}{2}(\overline{S}+a\widetilde{\sigma})\right]} \end{aligned} \tag{78}$$

We can realize that the exogenous liquidity risk is introduced as a linear combination of U_0, which simplifies notably the computations. Let

$$\frac{(1-h)}{\left[1-\frac{1}{2}(\overline{S}+a\widetilde{\sigma})\right]} = (1-h') \tag{79}$$

Then,

$$\frac{B_k\,(T)}{B_{k-1}\,(T)} \leqslant (1-l)(1-h') \tag{80}$$

which is very similar to expression (12). Therefore, we can use the results derived earlier, this is, the probability of incurring in a big loss larger than $\overline{L}$ in period k will be now:

$$P(k) = N(\frac{\ln((1-l)(1-h')) - \mu_k}{\sigma_k})\tau Q \tag{81}$$

where

$$h' = 1 - \frac{(1-h)}{\left[1 - \frac{1}{2}(\overline{S} + a\widetilde{\sigma})\right]} \tag{82}$$

In the same way we can extend the result to the case of a collateral formed with a portfolio of bonds. At $t = k\tau$ the possible loss will be equal to:

$$Loss_k = U_0 - \left\{ \alpha_{k-1} B_k(T_1) \left[1 - \frac{1}{2}(\overline{S}_1 + a\widetilde{\sigma}_1)\right] + q\alpha_{k-1} B_k(T_2) \left[1 - \frac{1}{2}(\overline{S}_2 + a\widetilde{\sigma}_2)\right] \right\} \tag{83}$$

or equivalently

$$Loss_k = U_0 - \frac{B_k(T_1) + q' B_k(T_2)}{(1-h_1')B_{k-1}(T_1) + (1-h_2')qB_{k-1}(T_2)} U_0 \tag{84}$$

where

$$\begin{aligned} 1 - h_1' &= \frac{1-h_1}{\left[1 - \frac{1}{2}(\overline{S}_1 + a\widetilde{\sigma}_1)\right]} \\ 1 - h_2' &= \frac{1-h_2}{\left[1 - \frac{1}{2}(\overline{S}_1 + a\widetilde{\sigma}_1)\right]} \\ q' &= q\frac{\left[1 - \frac{1}{2}(\overline{S}_2 + a\widetilde{\sigma}_2)\right]}{\left[1 - \frac{1}{2}(\overline{S}_1 + a\widetilde{\sigma}_1)\right]} \end{aligned} \tag{85}$$

and follow same methodology as we did.

6.3 Appendix 3: Non-Constant Marking to Market

In this part we will introduce the case where marking to market period is not constant. The reason to think about this possibility comes from the observation in real world. For instance, we can observe a weekend effect, i.e. although the marking to market is daily, during the weekend there are not margin calls, therefore, the marking to market period is not constant anymore. This new case does not affect our theoretical derivation, however, the effect on the probability of loss can be considerable.

Consider that the life of the contract can be divided in K periods t_k ($k = 1, 2, ...K$) where the interval τ_k is not fixed ($\tau_k = t_k - t_{k-1}$ is not constant). Then, the possible loss at time t_k can be written as:

$$loss_k = U_0 - \alpha_{k-1} B_k \tag{86}$$

or equivalently

$$loss_k = U_0 - \frac{U_0}{(1-h)} \frac{B_k}{B_{k-1}} \tag{87}$$

we know that

$$\ln(\frac{B_k}{B_{k-1}}) = \mu_k + \sigma_k z \tag{88}$$

where

$$\begin{aligned} & z \sim N(0,1) \\ \mu_k &= (m_k - m_{k-1}) + \frac{1 - e^{-a(t_k - t_{k-1})}}{a}(be^{-a(T-t_k)} + e^{-at_{k-1}}(r_0 - b)) \\ \sigma_k &= \sqrt{\sigma_{1k}^2 + \sigma_{2k}^2} \\ \sigma_{1k} &= \frac{1 - e^{-a(t_k - t_{k-1})}}{a}\sigma_r(\frac{1 - e^{-2at_{k-1}}}{2a})^{\frac{1}{2}} \\ \sigma_{2k} &= n_k \sigma_r (\frac{1 - e^{-2a(t_k - t_{k-1})}}{2a})^{\frac{1}{2}} \end{aligned} \tag{89}$$

Then, we have the following relationship between the determined maximum loss, $\overline{L} = lU_0$, and z.

$$\begin{aligned} lU_0 &\leq loss_k = U_0 - \frac{U_0}{1-h} e^{\mu_k + \sigma_k z} \\ &\Longrightarrow z \leq \frac{\ln((1-l)(1-h)) - \mu_k}{\sigma_k} \end{aligned} \tag{90}$$

Then,

$$P(loss_k \geq \overline{L}) = N\left(\frac{\ln((1-l)(1-h)) - \mu_k}{\sigma_k}\right)$$

The result is very similar to the case with constant marking to market.

6.4 Appendix 4: Default Following a Poisson Process

$$P(k) = \Pr(X_k \cap Y_k) = \Pr(X_k) \cdot \Pr(Y_k | X_k) \tag{92}$$

where

$$X_k = \{\frac{B_k(T)}{B_{k-1}(T)} \leq (1-l)(1-h)\} \tag{93}$$

$$Y_k = \{(k-1)\tau < \iota \leq k\tau\} \tag{94}$$

where $\Pr(X_k)$ has been already computed before. From (48), we compute the conditional probability at time 0,

$$\begin{aligned}
\Pr(Y_k \,|X_k) &= \Pr(k-1)\tau < \iota \leq k\tau \,|X_k) \\
&= \Pr(\tau > (k-1)\tau \,|X_k) - \Pr(\tau > k\tau \,|X_k) \\
&= E_0(e^{-\int_0^{(k-1)\tau}\lambda(s)ds} - e^{-\int_0^{k\tau}\lambda(s)ds} \,|X_k) \\
&= E_0(e^{-\int_0^{(k-1)\tau}\lambda(s)ds} \,|X_k) - E_0(e^{-\int_0^{k\tau}\lambda(s)ds} \,|X_k)
\end{aligned}$$

Notice that, in the first term, $e^{-\int_0^{(k-1)\tau}\lambda(s)ds}$ has nothing to do with X_k because $e^{-\int_0^{(k-1)\tau}\lambda(s)ds}$ and X_k happen at different times. So,

$$\begin{aligned}
E_0(e^{-\int_0^{(k-1)\tau}\lambda(s)ds} \,|X_k) &= E_0(e^{-\int_0^{(k-1)\tau}\lambda(s)ds}) \\
&= E_0(e^{-\int_0^{(k-1)\tau}(\lambda_s^o+\phi r_s)ds}) \\
&= E_0(e^{-\int_0^{(k-1)\tau}\lambda_s^o ds}) \cdot E_0(e^{-\int_0^{(k-1)\tau}\phi r_s ds})
\end{aligned}$$

On the other hand, $E_0(e^{-\int_0^{(k-1)\tau}\lambda_s^o ds})$ is similar to the pricing formula of a zero coupon bond maturing at time t_{k-1}. According to the Feynman-Kac theorem, $E_0(e^{-\int_0^{(k-1)\tau}\lambda_s^o ds})$ is the solution of a partial differential equation.

$$E_0(e^{-\int_0^{(k-1)\tau}\lambda_s^o ds}) = e^{m_\lambda(0,(k-1)\tau) - n_\lambda(0,(k-1)\tau)\lambda_0^0} \tag{95}$$

where

$$n_\lambda(t,T) = \frac{1-e^{-a_\lambda(T-t)}}{a_\lambda} \tag{96}$$

$$m_\lambda(t,T) = \frac{(n_\lambda(t,T) - T + t)(a_\lambda^2 b_\lambda - \frac{\sigma_\lambda^2}{2})}{a_\lambda^2} - \frac{\sigma_\lambda^2 n_\lambda^2(t,T)}{4a_\lambda} \tag{97}$$

From (53), we have

$$d(\phi r_t) = a'(b' - \phi r_t)dt + \sigma_r' dW_t \tag{98}$$

where

$$a' = a \tag{99}$$

$$b' = b\phi \tag{100}$$

$$\sigma_r' = \phi\sigma_r \tag{101}$$

Similarly,

$$E_0(e^{-\int_0^{(k-1)\tau} \phi r_s ds}) = e^{m'(0,(k-1)\tau)-n'(0,(k-1)\tau)\phi r_0} \tag{102}$$

where

$$n'(t,T) = \frac{1-e^{-a'(T-t)}}{a'} \tag{103}$$

$$m'(t,T) = \frac{(n'(t,T)-T+t)(a'^2 b' - \frac{\sigma_r'^2}{2})}{a'^2} - \frac{\sigma_r'^2 n'^2(t,T)}{4a'} \tag{104}$$

Therefore,

$$E_0(e^{-\int_0^{(k-1)\tau} \lambda(s)ds} | X_k) = e^{m_\lambda(0,(k-1)\tau)-n_\lambda(0,(k-1)\tau)\lambda_0^0}.e^{m'(0,(k-1)\tau)-n'(0,(k-1)\tau)\phi r_0} \tag{105}$$

Since λ_t^o is independent of default-free interest rate,

$$E_0(e^{-\int_0^{k\tau} \lambda(s)ds} | X_k) = E_0(e^{-\int_0^{k\tau} \lambda_s^o ds}) \cdot E_0(e^{-\int_0^{k\tau} \phi r_s ds} | X_k)$$

Again we have,

$$E_0(e^{-\int_0^{k\tau} \lambda_s^o ds}) = e^{m_\lambda(0,k\tau)-n_\lambda(0,k\tau)\lambda_0^0} \tag{106}$$

Define a savings account

$$B(t) = e^{\int_0^t r_s ds} \tag{107}$$

Under risk-neutral probability measure, the relative price $\frac{B(t,T)}{B(t)}$ is a martingale, and follows

$$d\left(\frac{B(t,k\tau)}{B(t)}\right) = \frac{B(t,k\tau)}{B(t)}\left(-\sigma_r n(t,k\tau)\right) dW_t \tag{108}$$

So

$$\frac{B(k\tau,k\tau)}{B(k\tau)} = \frac{B(0,k\tau)}{B(0)} \exp(\int_0^{k\tau} -\sigma_r n(t,k\tau) dW_s - \frac{1}{2}\int_0^{k\tau} \left(-\sigma_r n(t,k\tau)\right)^2 ds) \tag{109}$$

Since $B(0) = 1, B(t_k, t_k) = 1$,

$$\frac{1}{B(k\tau)B(0,k\tau)} = \exp(\int_0^{k\tau} -\sigma_r n(t,k\tau) dW_s - \frac{1}{2}\int_0^{k\tau} \left(-\sigma_r n(t,k\tau)\right)^2 ds) \tag{110}$$

$$\begin{aligned}\frac{1}{B(k\tau)^\phi B(0,k\tau)^\phi} &= \exp(\int_0^{k\tau} -\phi\sigma_r n(s,k\tau) dW_s - \frac{1}{2}\int_0^{k\tau} \phi\left(-\sigma_r n(s,k\tau)\right)^2 ds) \\ &= \xi(k\tau)\exp(\frac{1}{2}\int_0^{k\tau} \phi(\phi-1)\sigma_r^2 n^2(s,k\tau) ds)\end{aligned}$$

where

$$\xi\left(k\tau\right)=\exp(\int_{0}^{k\tau}-\phi\sigma_{r}n(s,k\tau)dW_{s}-\frac{1}{2}\int_{0}^{k\tau}\left(-\phi\sigma_{r}n(s,k\tau)\right)^{2}ds) \tag{111}$$

Hence, we have

$$\begin{aligned} E_0(e^{-\int_0^{k\tau}\phi r_s ds}|X_k) &= \frac{1}{\Pr(X_k)}E_0(e^{-\int_0^{k\tau}\phi r_s ds}\mathbf{1}_{X_k}) \\ &= \frac{B(0,k\tau)^{\phi}}{\Pr(X_k)}E_0(\frac{1}{B(k\tau)^{\phi}B(0,k\tau)^{\phi}}\mathbf{1}_{X_k}) \\ &= \frac{B(0,k\tau)^{\phi}}{\Pr(X_k)}E_0(\xi\left(k\tau\right)\mathbf{1}_{X_k})\exp(\frac{1}{2}\int_0^{k\tau}\phi(\phi-1)\sigma_r^2n^2(s,k\tau)ds) \end{aligned}$$

It can be shown that

$$B(0,k\tau)^{\phi}\exp(\frac{1}{2}\int_{0}^{k\tau}\phi(\phi-1)\sigma_{r}^{2}n^{2}(s,k\tau)ds)=e^{m'(0,k\tau)-n'(0,k\tau)\phi r_{0}} \tag{112}$$

$m'\left(0,t_k\right)$ and $n'\left(0,t_k\right)$ are defined in (103) and (104).
Define a new probability measure P^{t_k} :

$$P^{k\tau}\left(A\right)=E(\xi\left(k\tau\right)\mathbf{1}_{X_k}),\quad \forall A\in\mathbf{F}_{k\tau} \tag{113}$$

Under P^{t_k},

$$W_t^{k\tau}=W_t+\phi\sigma_r\int_0^t n(s,k\tau)ds \tag{114}$$

is a Brownian motion. The default-free interest rate under the new probability measure follows:

$$dr_t=a(b-\frac{\phi\sigma_r^2}{a}n(t,k\tau)-r_t)dt+\sigma_r dW_t^{k\tau} \tag{115}$$

Similarly,

$$E_0(\xi\left(k\tau\right)\mathbf{1}_{X_k})=P^{k\tau}\left(X_k\right)=N(\frac{\ln(1-l)(1-h)/\left(1-\theta\right)-\mu_k'}{\sigma_k'})$$

where

$$
\begin{aligned}
\mu_k' &= (m(k\tau+\delta,T)-m_{k-1}(T)) + \frac{1-e^{-a(\tau_k+\delta)}}{a}(be^{-a(T-k\tau-\delta)}+e^{-a(k-1)\tau}(r_0-b)) \\
&\quad -n\,(k\tau+\delta,T)\left(\eta\,(k\tau+\delta,t_k+\delta)-\eta\,((k-1)\tau,k\tau+\delta)\,e^{-a(\tau_k+\delta}\right) \\
\sigma_k' &= \sqrt{\left(\frac{1-e^{-a(\tau_k+\delta)}}{a}\sigma_r\right)^2(\frac{1-e^{-2a(k-1)\tau}}{2a})+(n(k\tau+\delta,T)\sigma_r)^2(\frac{1-e^{-2a(\tau_k+\delta)}}{2a})} \\
\eta\,(s,t) &= \frac{\phi\sigma_r^2}{a^2}\left[\frac{1}{2}e^{-a(t-s)}(1-e^{-2as})-(1-e^{-as})\right] \qquad (117)
\end{aligned}
$$

where δ is time to capture, and θ is the percentage representing liquidity. Hence,

$$
E_0(e^{-\int_0^{k\tau}\lambda(s)ds}\,|X_k) = e^{m_\lambda(0,k\tau)-n_\lambda(0,k\tau)\lambda_0^0}\cdot e^{m'(0,k\tau)-n'(0,k\tau)\phi r_0}\cdot\frac{P^{k\tau}\,(X_k)}{\Pr(X_k)} \qquad (118)
$$

$$
\begin{aligned}
\Pr(Y_k\,|X_k) &= e^{m_\lambda(0,(k-1)\tau)-n_\lambda(0,(k-1)\tau)\lambda_0^0}\cdot e^{m'(0,(k-1)\tau)-n'(0,t_{k-1})\phi r_0} \qquad (119)\\
&\quad -e^{m_\lambda(0,k\tau)-n_\lambda(0,k\tau)\lambda_0^0}\cdot e^{m'(0,k\tau)-n'(0,k\tau)\phi r_0}\cdot\frac{P^{k\tau}\,(X_k)}{\Pr(X_k)}
\end{aligned}
$$

When $\phi=0$, there is no correlation between the interest rate and the intensity of default, therefore we have

$$
\Pr(Y_k\,|X_k) = e^{m_\lambda(0,(k-1)\tau)-n_\lambda(0,(k-1)\tau)\lambda_0^0} - e^{m_\lambda(0,k\tau)-n_\lambda(0,k\tau)\lambda_0^0} \qquad (120)
$$

The simplest case is that the intensity of the default process is a constant λ, in that case, $\Pr(Y_k\,|X_k)$ can be written as:

$$
\begin{aligned}
\Pr(Y_k\,|X_k) &= E_0(e^{-\int_0^{(k-1)\tau}\lambda(s)ds}\,|X_k) - E_0(e^{-\int_0^{k\tau}\lambda(s)ds}\,|X_k) \\
&= e^{-\lambda(k-1)\tau}(1-e^{-\lambda k\tau}) \qquad (121)
\end{aligned}
$$

Denote the probability that default will happen within one year,

$$
Q = 1-e^{-\lambda} \qquad (122)
$$

Then

$$
\begin{aligned}
\Pr(Y_k\,|X_k) &= (1-Q)^{\iota_1+\iota_2+\ldots+\iota_{k-1}}(1-(1-Q)^{k\iota}) \\
&\approx \prod_{i=1}^{k-1}(1-Q\iota_i)\cdot Q\iota_k \qquad (123)
\end{aligned}
$$

(123) is a very good approximation for high frequent marking to market.
Finally we have that the total probability of having a loss larger than $\overline{L}$ is:

$$P = \sum P(k) \tag{124}$$

where

$$\begin{aligned} P(k) &= e^{m_\lambda(0,(k-1)\tau)-n_\lambda(0,(k-1)\tau)\lambda_0^0} \cdot e^{m'(0,(k-1)\tau)-n'(0,(k-1)\tau)\phi r_0} \cdot \Pr(X_k) \\ &\quad -e^{m_\lambda(0,k\tau)-n_\lambda(0,k\tau)\lambda_0^0} \cdot e^{m'(0,k\tau)-n'(0,k\tau)\phi r_0} \cdot P^{k\tau}(X_k) \end{aligned} \tag{125}$$

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