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Serial sovereign default: the role of shocks and fiscal habits

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Abstract

We confront five stylized facts related to sovereign default: 1) the presence of serial defaulters; 2) the prevalence of partial over complete default; 3) the counter-cyclical of default; 4) non-linearity of sovereign spreads; and 5) heterogeneous outcomes among serial defaulters. In a model that integrates fiscal uncertainty and habit formation in policy, assuming incomplete financial markets, we demonstrate that default is habit and shock driven as well as non-strategic and involuntary. Moreover, there is no requirement for sanctions to sustain trading. In spite of dealing with serial defaulters, partial default is a robust equilibrium. We characterize good and bad fiscal habits and, that with the latter, expected default increases with habit persistence. The impact of habits on the expected default rate is the opposite of its effect on both the interest rate on public debt and base interest of the economy. The presence of habits also has implications for the cost of debt, default risk premium and the cost of default, and can shed light on country heterogeneities.

Keywords: Default Probability; Habit Formation; Fiscal shocks; Topological Robustness.

JEL Codes: G18; H63

Non Technical Summary

This paper introduces a new perspective on sovereign debt default. We construct a two-period general equilibrium model with a government and representative lender. The borrower is a country which develops habit formation with respect to its primary surpluses since actual surpluses systematically differ from planned surpluses. As lenders recognize this type of borrower, fiscal habits influence the asset market clearing condition. The expected default rate depends on the financial market conditions, on the interaction between the lender and the government, the interest rate on the public debt, fiscal shocks, and on uncertainty. However, surprisingly, this probability is not necessarily equal to one. The impact of habits on the expected default rate is the opposite of its impact on both the interest rate of the public debt and the base interest rate of the economy. Its impact on the probability of default depends on the evolution of habits.

The presence of sovereign habits can modify the effect of fiscal shocks and impact the probability of default, the prevalence of partial default outcomes and movements in default premia. We also distinguish between ‘good’ and ‘bad’ fiscal habits. We introduce a new perspective on repeated sovereign default. We construct a 2 two-period general equilibrium model with uncertainty in the second period. We demonstrate that, despite such uncertainty, partial default is a robust equilibrium outcome. In addition to volatility, the sovereign’s own “fiscal habits” can also alter the probability and characteristics of default. In that respect, we also draw a distinction between good and bad fiscal habit regimes. These mechanisms shed light on the phenomenon of serial sovereign default.

Throughout history, governments have demonstrated that “serial default” is the rule, not the exception ... the fact that sovereign defaults tend to recur like clockwork in some countries, while being absent in others, suggests that there must be a significant explainable component

Reinhart and Rogoff (2004)

1 Introduction

This paper introduces a new perspective on sovereign default. Our starting point, in line with the quote above, is the observation that some sovereigns are serial defaulters. That is to say, they default with some marked regularity.¹ For example, Argentina, Bolivia, Mexico, Nigeria, Peru, and Venezuela have spent much of the last decades in periods of default or protracted re-negotiations (Eichen, 2003; Enderlein et al., 2012; Meyer et al., 2019).

Serial default poses a challenge for the literature since, given the postulated costs of default (e.g., sanctions and reputational damage), one might instead expect default (including partial default) to be a rare and extraordinary outcome.² The fact that it is not, begs the question of why some sovereigns, knowing the costs, not only do default, but habitually default. Likewise, why do rational capital markets not only grant access, but then re-establish access to such repeat offenders (see Kehoe and Levine, 2008; Gelos et al. 2011).

Our paper addresses these issues in a tractable theoretical framework. In so doing, we try to confront several stylized facts. A first fact is that, under severe fiscal stress, partial default is a vastly more common outcome than complete default (or repudiation), see Arellano et al. (2019). This is worth emphasizing since much of the theoretical literature has instead framed default in binary terms. Our model (described in subsequent sections), however, demonstrates that default can not only occur in equilibrium but that partial default is a stable, robust outcome. This in itself constitutes an important stepping-stone in understanding serial default, since it places the commonplace nature of default at the heart of our analysis. Moreover, our framework can also be informative about which model elements either expand or contract the available space for partial default to occur.

A second, more straightforward stylized fact is that sovereign default tends to be counter-cyclical:³

¹ The typical definition of default (as for example followed by rating agencies) being a situation of failure to meet the principal or interest payment on the due date.

² Influential studies in the field include Eaton and Gersovitz (1981), Arellano (2008), Panizza et al. (2009), Reinhart and Rogoff (2011), Mendoza and Yue (2012). Stähler (2013) and Aguiar and Amador (2014) provide excellent discussions and overviews.

³ By contrast, if defaults were voluntary, they may be more likely to be pro-cyclical. Indeed, much of the earlier default literature

sovereigns mostly default in times of weak or volatile growth (see Tomz and Wright, 2007; Mendoza and Yue, 2012; Aguiar and Amador, 2014).⁴ In line with that, many ‘serial defaulters’ are Emerging nations whose economies are precisely known to be subject to substantial volatility reflecting external economic shocks and dependency on often erratic short-term capital. In Argentina’s case, for example, although it has experienced periods of largely unbroken growth (e.g., 2003-2011), there have also been episodes of severe swings in activity (e.g., 2011 onward).⁵

Third, even if we can rationalize partial default and the primacy of volatile economic shocks in the default process, this still leaves open the issue of why certain sovereigns are, or may be, serial defaulters. To that end, we additionally explore the idea that sovereigns (much like individuals) are prone to “fiscal habit formation”. Consider the following argument. A sustainable public debt is one which does not exceed the present value of expected primary surpluses (i.e., revenues minus expenditures). However, in the presence of volatile economic shocks, the government may not have sufficient control over its primary surplus. Accordingly, actual and planned surpluses may diverge. If this process is repeated frequently it may become habit forming: in effect, past primary surpluses that generally fall short of planned ones form a *stock* of fiscal habits. As lenders learn how to deal with this type of borrower, habits influence the asset market-clearing condition. The presence of fiscal habits, moreover, can amplify or curtail the effects of shocks on primary surpluses and, in turn, alter the probability and characteristics of default in several interesting ways, without overturning our first result on the robustness of the partial-default outcome per se.

The aspects of volatility and habits buttresses the remaining facts of interest: the non-linearity of spreads and heterogeneities among serial defaulters. Sovereign spreads can be highly volatile and non linear. Our framework speaks to that since the presence of even a simple form of habit formation generates a non-linearity in the transmission of fiscal outcomes to spreads. This, in turn, speaks to country differences, since we know that countries with similar fiscal profiles can have different outcomes in terms of default probabilities and spread experiences, this we attribute to the nature of their habit formation (‘good’ or ‘bad’ habits).

predicted pro-cyclical default rates (assuming the costs of autarky are low in a boom), e.g., Kehoe and Levine (1993). Countercyclical default however is also true at the level of commercial lenders. This can for example be gauged by looking at the delinquency rate on credit card loans: see St. Louis Fed FRED data series DRCLACBS.

⁴ In line with this, the literature distinguishes between “willingness to pay” and “capacity to pay”. For example, in its default in 2008, Ecuador’s is generally seen as belonging to the former case given the Correa government’s classification of the existing debt as “odious”.

⁵ Aguiar and Gopinath (2007) show that the average output volatility in emerging economies is twice that of developed economies (in some countries, like Argentina, that ratio is even higher). See also our later figure 3.

We address these issues in a general equilibrium model with a representative country (sovereign) and a lender. As in models of securities trading, there are two periods 0 and 1, and $S > 1$ possible states of nature in period 1, e.g., Geanakoplos and Zame (2014). At period 0, the sovereign and lender interact, and between 0 and 1, the state of nature is revealed, sovereign debts are resolved.⁶

Our analysis is carried out into two stages. First, we assume that the initial primary surplus is impacted by exogenous fiscal shocks. These shocks follow a probability distribution such that the value of expected, planned primary surplus of the second period is always strictly positive, although there are states (of nature) where the primary surplus is negative.⁷ This is key to guarantee continued trading between the lender and government. We analyze how shocks affect repayment and the probability of default (be it complete, partial default, or no default) and the cost of debt. The default regime that arises turns out to depend on the relation between the government's leverage (its indebtedness) and the size of fiscal shocks. For instance, if the government had a moderate (or bounded) leverage, we would witness a partial default. However, if leverage exceeds the size of shock, a complete default results. By implication, therefore, default in our framework matches the first two stylized facts, being essentially involuntary and counter-cyclical.

Although only two-periods, the intertemporal nature of our model can still capture important dynamic stylized facts (for example, implicitly, the prevalence of partial default followed by re-negotiation).⁸ An additional important remark is that the model is not a game, even if we use the generalized games methodology to demonstrate the existence of equilibrium. It is further a general equilibrium model with incomplete markets where the government only enters the economy, (implicitly or explicitly) making a 'fiscal effort' to balance its budget. However, that same fiscal effort begets fiscal habits which affect the whole economy as well as default outcomes themselves.

In the second stage of our treatment, accordingly, we make that fiscal correction *endogenous* via habit formation. For that, we assume that the events that lead to fluctuations in the primary surplus are affected by the government's own fiscal efforts. By "effort", we mean any conduct the government (its

⁶ We are implicitly assuming a private lender here, although in practice sovereigns often also deal with multinational intergovernmental institutions like the IMF, the World Bank or the 'Troika' with potentially soft budget constraints, macroeconomic conditionality, and a more political-economy perspective (e.g., IMF, 2012; Lane, 2012; Bournakis et al., 2019).

⁷ Negative primary surplus have recently been considered by Brunnermeier et al. (2020) to explain why countries with persistently negative primary surpluses can have a positively valued currency and low inflation.

⁸ Our model has been developed in two periods since we are assuming that the future primary surplus is not large enough to exceed the face value of the public debt. However, if this were not the case, we would need a multi-period or infinite-horizon model where the government would transfer either surpluses or deficits to its successors. This will go in the direction of a theory on the public leverage cycle.

politicians, rulers) adopts to modify its primary balance.⁹ After this analysis, we determine how the parameter measuring habits affects variables of interest. Among other things, we demonstrate that, under “bad” fiscal habits, the lower will be the expected value of the primary surplus and the higher the default probability (where, for now, bad habits can be loosely defined as authorities placing weaker effort on fiscal stabilization relative to the size of the fiscal shock). The presence of habits also has implications for the cost of debt, the external risk premium and the cost of default. These various elements provide us with a mechanism to explain why certain sovereigns find themselves in default on a quite regular basis, thus speaking to the third of our stated stylized facts.

Note, since this is a two-period model, there need be no presumption that the sovereign is infinitely lived (or dynastic) such that fiscal habits are *always* present. Our interest rather is to assess whether a sovereign that experiences fiscal stress and a shock magnified by its own fiscal habits, can be shown to have a material bearing on default characteristics. Indeed, habits and their impact on shock propagation, has implications for how we view the default process: it offers a middle course between “bad-luck” involuntary default and strong assumptions about governments’ ability to voluntarily default (i.e., making complex welfare comparisons across default regimes).

Organization Section 2 compares and contrasts our approaches to the existing literature. Section 3 provides some brief historical motivation, establishing the prevalence of default episodes and the characterization of some sovereigns as serial defaulters: having repeated default episodes, and highly volatile economies. Section 4 presents the model, the agents and defines the characterization of equilibrium. Section 5 establishes our main results. The first has to do with the existence of equilibrium across default, no-default and partial default regimes and the non-triviality and robustness of equilibrium outcomes. We additionally examine the effects of economic disturbances and boom probabilities on the cost of debt and the likelihood of default. In Section 6 we introduce the idea of fiscal habits, making the distinction between good and bad habits. Habits imply that the government follow an inertial rule that weighs the effect of past shocks and past fiscal efforts. We show that habits also have a bearing on default characteristics, and can magnify or attenuate our previous channels. Section 7 determines the factors which influence the sovereign default risk premium, which is the additional amount that a government must pay to compensate a lender for assuming default risk, as well as the cost of default. Section 9 makes some general remarks and caveats, and summarizes and our main results. Section 10 concludes and

⁹ See Arellano et al (2015) for an analysis of the role of fiscal flexibility in external debt crises, comparing US states and Canadian provinces as against European nations.

discusses future research directions. Additional material appears in the appendices.

2 Related Literature

Our paper is related to the vast literature on debt crises (e.g., Bulow and Rogoff, 1989; Diamond and Rajan, 2001; Jeanne, 2009; Reinhart and Rogoff, 2009, Ch 4) and on sovereign default in general equilibrium (e.g., Arellano, 2008; Mendoza and Yue, 2012) – in the sense of having both commonalities, and points of departure from that literature.

Arellano's seminal paper is more complex in temporal aspects and includes a productive sector. We share the incompleteness of financial markets, which is key to guarantee a positive probability of default in equilibrium. Our framework however demonstrates that, if there is default, partial default is more robust than complete default. Partial default has been examined by Arellano et al. (2019) and Arellano et al. (2020). Moreover, like Besancenot et al. (2004) we assume that the government suffers fiscal shocks on its primary surplus although, unlike them, we consider that the government forms habits with respect to the reference levels of shocks (see below).

Another stream of literature with which our study is related is where there is default in equilibrium with incomplete markets, as pioneered by Dubey et al. (2005). That paper includes an fictitious agent (interpreted as the government) whose presence in the economy guarantees the existence of a non-trivial equilibrium. More precisely, this fictitious agent boosts the economy by trading assets in small-enough amounts and without defaulting. Our paper is also related to Geanakoplos (2010) on how to determine the interest rate in times of crisis. Following that paper, ours determines the interest rate and the probability of default in a unique equation in times of crisis, which we define to be an event where the government does not pay its debts. As we shall see in the next section, counter-cyclical sovereign spreads are a well-known stylized fact of sovereign default, Aguiar and Gopinath (2006).

Notwithstanding, our approach has elements which are also distinct from the prevailing sovereign-default literature. The more standard literature, as in Arellano (2008), allows for voluntary default: i.e., the sovereign can choose to default and suffer the consequences even if it can repay. Indeed, in an earlier contribution, Bulow and Rogoff (1989) suggest that a direct sanction is necessary to support any international borrowing and lending, otherwise the government has an incentive to default too much. If there is no direct sanction, trading may still take place with no equilibrium default but in that case the government faces a default premium and borrows too little. In our treatment, by contrast, even though it

is understood that default may exist in equilibrium, there is no necessity for sanctions to sustain trading. Moreover, in our model, default is largely involuntary and non strategic, being driven by the impact of sufficiently large shocks and habits.

A last departure point from the default literature, and particular contribution of our paper, is aforementioned the integration of habits in policy. The habit-formation hypothesis has a long and distinguished lineage in economics.¹⁰ As far as we know, though, habit formation has yet to be explicitly and formally linked to policy actions, and, in turn, matters of serial sovereign default. Notwithstanding, it has long been understood in the political-economy literature that governments are composed of individuals, rulers, or a particular political or bureaucratic class with their own inertial preferences, objectives and defined rules-of-the-game (Buchanan and Tullock, 1962, being a classic text). Other papers which have also recognized the importance of fiscal rigidities and fiscal preferences in the budgetary and developmental process include Munoz and Olaberria (2019) and Herrera and Olaberria (2020).

More specifically, fiscal habits could thus be justified with respect to standard political-economy, deficit-bias and common-pool arguments (Alesina and Tabellini, 1990); or the effect of protracted political uncertainty on the likelihood of defaults (Cuadra and Saprizza, 2008). Habits could also be a function of regime duration: political longevity may ingrain expectations of fiscal stress and build expertise in the management of that stress. For instance, in Argentina's case, the *Partido Justicialista* have dominated congressional and presidential politics over the last four decades.¹¹ Another means to justify fiscal habit formation relates to the differences between rules and discretion. Fiscal rules are designed to solve the solvency and credibility problems that plague some countries, while habits reflect a type of inertial force based on policy sluggishness, i.e., if countries lack political or other incentives to fix imbalances, they tend to accommodate in an idle way to the usual way of doing business, which is essentially habit formation.¹²

Many existing papers have emphasized dysfunctional political-economy factors as contributing to

¹⁰ For example in explaining consumption patterns (e.g. Duesenberry, 1949; Brown, 1952; Alessie and Lusardi, 1997), addressing the equity premium puzzle (Constantinides, 1990), monetary policy transmission (Fuhrer, 2000; Faria and McAdam, 2013), addiction patterns (Dockner and Feichtinger, 1993), growth (Carroll et al., 2000; Gómez, 2015), and job satisfaction (Clark, 1999). Veblen (1904) gave perhaps the first fully-formed definition of habits in the outlook and behaviors of workers and capitalists.

¹¹ Moreover, particularly in South America, sovereign defaults often cluster in time (e.g., the early 1980s), reflecting common correlated shocks and economic spillovers. An open question however is whether habits in one country in an area with high economic integration can – through its magnification of the initial shock and spillover to neighbors – help explain such local clustering, even with idiosyncratic country shocks. A likely example of the latter is the banking crisis in Uruguay in the early 2000s reflecting the freezing of bank accounts in neighboring Argentina.

¹² Historical examples of ingrained perspectives on the role of fiscal policy include, perhaps most famously, the “Treasury View”, first formally attributed to Hawtrey (1925) and attached by Keynes in the *General Theory*. In German political economy the specter of the Swabian housewife plays a similar role .

sovereign default (Reinhart et al. 2003; Kohlscheen, 2007; Cuadra and Sapriz, 2008; Rijkkeghem and Weder, 2009). Unlike ours, though, these are empirical papers with quite distinct results and identification strengths.¹³ Moreover, one of these papers (Reinhart et al., 2003) defined “debt intolerance” as a feature of serial defaulters: namely, that countries with a past default history find they can only sustain relatively low levels of debt.¹⁴ In an important recent paper, Amador and Phelan (2021) also emphasized the reputation of the serial defaulter and how they might graduate into the club of debt tolerant sovereigns after a sufficient period of non-default (see also Levine, 2021, on this theme).

Given this background, we demonstrate that formalizing policy habit-formation shifts the probability and nature of sovereign default in several interesting ways. Our analysis also allows us draw to a distinction between ‘good’ and ‘bad’ fiscal habits: each with their own default implications and comparative statics.¹⁵

3 Some Brief Historical Motivation

Before we come to the theoretical framework, we can place our themes in some broad recent historical perspective. This allows us to emphasize the following:¹⁶

- i) Sovereign default events are not especially rare;
- ii) Some sovereigns are repeat or serial defaulters;

Among these “serial defaulters”, we additionally observe:

- iii) Sovereign deficit ratios are not necessarily excessive in size;
- iv) Sovereign defaulters tend to have relatively volatile economies, and;
- v) Sovereign spreads behave in a highly volatile and non linear manner, with marked country heterogeneity.

¹³ Cohen and Valadier (2011), noted that few of the past serial defaulters had a credit event around the Great Recession, thus holding out the possibility that fiscal policy makers had learned from previous crises.

¹⁴ For a discussion of this literature see Eichengreen (2007) who advocates the “original sin”, argument that some countries cannot solely issue bonds in their own currencies, leading to a higher risk premium and default probability.

¹⁵ Alfaro and Kanczuk (2005) discuss the difference between a ‘good’ and a ‘bad’ sovereign, although this is very different from our approach. A “good” sovereign places high importance on future consumption and discounts it at the same rate as households, whereas a “bad” sovereign emphasizes current consumption and never services its debt.

¹⁶ See also Asonuma (2016) for an analysis of stylized facts for serial defaulters including persistence in default patterns and borrowing costs.

Consider the two panels of Figure 1. These use the modern data set of Asonuma and Trebesch (2016). They consider periods which registered the start of one or more default or restructuring events, either as a default or announcement of default. Panel A considers all countries in their database (just under 80 countries, for just under 200 events). In almost all years of the sample there was some default event, with the 1980s witnessing a marked spike.¹⁷ That default is not so uncommon is summed up well by the detection of 24 such events in 1983 alone.

In Panel B we choose to isolate countries that have at least five or more default events.^{18, 19} These modern, serial defaulters are mostly but not exclusively Latin American countries. To illustrate: Jamaica had several default events in quick succession (77-78, 80, 83-84, 86-87). Likewise, observe Argentina (82, 85, 88, then later, 01, 19); Brazil (82-84, 86, 89); Ecuador (82-84, 86, then later, 99, 08, 20). The duration of these default episodes can sometimes be over a decade long (e.g., Uribe and Schmitt-Grohé, 2017, table 13.1). Moreover, these staggered, largely adjacent defaults arguably constitute a more pragmatic interpretation of serial default than one based on events scattered diffusely over history (even if involving the same country). Nearby default events may also make for a more cogent case for the importance of fiscal habits. Naturally, default episodes compromise other policy targets link to budgetary and demand matters. For instance, over 1980–2019, the mean consumer price inflation rate was 215% and 309% per year for Argentina and Brazil, respectively, for a mean unemployment rate of around 10-11% for both.²⁰

Figure 2 plots primary deficits as a percentage of GDP over 1990-2020 for the identified countries. Although there is some volatility, it is also clear that there is a great deal of persistence over time. This is certainly suggestive evidence that there are likely to be habit-like mechanisms within the evolution of primary deficits for countries in general, and equally so for countries which have reported repeated default events. What is also noticeable is that these primary deficits (compared to those of, say, the G10 nations) are not especially large, often of the order of around 5% of output. This is consistent with the “debt intolerance” interpretation. Figure 3 benchmarks the selected serial defaulters on US real GDP growth. The gulf in terms of volatilities compared to the US is readily apparent.

Finally, Figure 4 shows the sovereign spreads (in basis points) of our selected serial defaulters in two

¹⁷ See the original source for information on when and how these default events were concluded, and their financial scale.

¹⁸ Unlike most other studies, Asonuma-Trebesch is monthly and thus they can code intra-year default events. This can be appreciated from the right hand graph where we have several such events: e.g., Nigeria with two events in 1982, and Ukraine with two events in 2015.

¹⁹ Meyer et al. (2019) have a different classification of serial defaulters: 60 countries that have defaulted on their external debt at least twice since 1815 and/or who were in default for protracted spells, defined as a share of years in default above the sample median.

²⁰ Source: unemployment (IMF WEO database April 2021 vintage), inflation (<https://www.worlddata.info/>). All data derived from official domestic sources.

panels, partitioned by average size. All countries have at times experienced huge and persistent increases in spreads reflecting economic, political and fiscal stress. Moreover, while there is often commonality in the qualitative pattern of those spreads, there is also marked country heterogeneity: some countries impacted by shocks recover their access and borrowing terms at different speeds than others. Further the substantial volatility and non-linearity of the spreads is apparent, again marked by cross-country heterogeneity (for a discussion see Aguiar et al. 2016). These figures help set the scene for our analysis given that we shall discuss the deficits at which countries default and the resulting schedule of their spreads and premia.

4 The Model

We consider a two-period model with uncertainty only in the second period modeled by a finite set S states of the nature (e.g., economic boom and busts). The economy has only a single good available in each period and in each state of nature so that the commodity space is $\mathbb{R}_+^{1+S} = \mathbb{R}_+ \times \mathbb{R}_+^S$. We assume that this good is the numeraire and all other later-defined variables will be measured in terms of this good.

We assume two agents: a representative lender and a government (or sovereign). The lender is characterized by utility function $u : \mathbb{R}_+^{1+S} \rightarrow \mathbb{R}$ and her initial endowment $W \in \mathbb{R}_+^{1+S}$. The lender and the government interact via financial markets. The government issues bonds to fund its expenditure. Taxes $T \in \mathbb{R}_+^{1+S}$ and expenditures $G \in \mathbb{R}_+^{1+S}$ are taken to be exogenous as in Gale (1990), and we compact them into a single variable: the primary surplus $E = (T - G)$.²¹ Thus, the government is characterized by its primary surplus stream $E \in \mathbb{R}^{1+S}$. In order to not accrue indexes, a typical element z belonging to $\mathbb{R}^{1+S}$ will be represented by a vector $(z_0, z_1, \dots, z_S)$ where z_0 refer to the present and $z_s, s = 1, \dots, S$, refers to the future. Accordingly, if we write z_s , it will be understood that s runs from 1 to S .

4.1 Assets and Default

There is only a short-lived (i.e., 1 period) security which is assumed to be a non-contingent bond. If q denotes its price, then the interest rate r is defined by $q = (1 + r)^{-1}$, which will be determined in equilibrium. Note that although the security is non-contingent (i.e., it pays the same quantity in all states of nature), this security is subject to default risk, since our model allows the government to default on

²¹ For an analysis of the potentially time varying effect of different timing and compositions of fiscal changes on the economy, see Bi et al. (2013).

the bonds issued.²² Specifically, the government could default if the value of the second-period primary surplus is impacted by fiscal shocks. Our economy is therefore represented by array $\mathcal{E} = [(u, w); E]$ which consists of a representative lender and a government.

4.2 The Government and Lender Problems

The government's problem is to issue debt $\varphi \geq 0$. The variable $D \in \mathbb{R}_+^S$ is a choice variable of the government and denotes the amount it repays from the debt incurred in the first period.²³ If D is below the face value (or the claim) of the bond, the government is in default. Accordingly, we have the following set of equations:

$$0 = E_0 + q\varphi, \quad (1)$$

$$D_s = E_s^+ \quad (2)$$

$$0 \leq D_s \leq \varphi \quad (3)$$

Equation (1) shows that, in the first period, the government issues debt to fund its deficit $E_0 < 0$. Equivalently, $\varphi = -E_0/q = -E_0(1+r)$. Equation (2) states the government finances its payments using the positive part²⁴ of its primary surplus $E_s, s = 1, \dots, S$. Inequality (3) makes clear that the government is allowed to default. That is, the government pays D_s less than or equal to what it must repay (φ) on the debt $q\varphi$ incurred in the first period.

Let us denote by Γ the set of all states of nature such that $E_s > 0$. According to (2), the government repays some part of its debt in such states. Thus in states $s \in \Gamma$ there will never be a complete default. However, in the states belonging to the complement of Γ , denoted Γ' , the sovereign will always be in complete default. That is, $D_s = 0, \forall s \in \Gamma'$. So, henceforth, by convention, we shall say that the government is in default if $\exists s \in \Gamma : 0 < D_s = E_s < \varphi$. Otherwise, the government does not default. That is, $0 < D_s = E_s = \varphi, \forall s \in \Gamma$.

²² The default mechanism analyzed in this paper is similar to that of collateralized loans, see for instance, Geanakoplos and Zame (2014). In the latter, there is a default if the value of the collateral falls below the debt. In that case, the collateral is transferred smoothly or without resistance to the lender. In our model, the default occurs if the future surplus falls below the debt and, like the collateral models, this surplus is transferred to the lender as established in the debt contract. Notice, however, the lender cannot confiscate the future surplus as they can in collateral models.

²³ In our setup, there is no necessity to define whether the debt is, or is not, external (e.g., debt issued in a foreign currency, held abroad etc.). And, being a two-period model, the question of debt maturity structure does not arise.

²⁴ For any real number m its positive part, denoted by m^+ is, by definition, given by $\max\{0, m\}$.

Note it is implicitly assumed that the future primary surplus did not exceed the claim of the public debt, see inequality (3). However, it is useful to note that what, on one hand, the government actually decides to repay depends on the value of the second-period primary surplus. In other words, conditional on the shock and habit formation, default is not voluntary. We do not allow the government to pay anything other than E_s . That is, it repays what nature dictates. On the other hand, what (3) essentially tells us is that the value of the E_s is at most the face value of the public debt incurred. So, at a fundamental level, what is implicit is that the future primary surplus is not sufficiently large (and could even be negative), but in good times it will be enough to pay off some or all of the debt.

The lender's problem by contrast is to choose a consumption plan $c \in \mathbb{R}_+^{1+S}$ and to issue $\theta \geq 0$ in order to maximize $u(c)$ subject to the two following budget constraints:

$$c_0 + q\theta \leq w_0 \quad (4)$$

$$c_s \leq w_s + (1 - \pi_s)\theta \quad (5)$$

The first states that the representative lender spends on consumption and the purchase of public debt, financed by the first-period initial endowment. The second-period budget constraint (5) shows that consumption is funded by their s -state contingent initial endowments and by asset receipts $(1 - \pi_s)\theta$. The factor $1 - \pi_s$ is called the payment (or delivery rate) which is defined in terms of the default rate π_s .

Remark 1 *Like the government in which the primary surplus E incorporates expenditures and taxes, we can think of the lender's initial endowment in (4) and (5) as containing taxes but we do not make this explicit since our main focus is not to study the impact of taxes per se, but rather the fiscal shocks that could affect E .*

4.3 Equilibrium

We shall now define the equilibrium concept in a habitual manner. An equilibrium for this economy consists of optimal choices both for the lender and government, wherein all markets clear. Moreover, like Dubey et al. (2005), we assume that lenders have the capacity to anticipate future rates of default. Default is then consistent with the orderly functioning of markets. We formalize this in the following definition.

Definition 1 *An equilibrium for this economy $\mathcal{E}$ consists of an asset price and default rates (q, π) and a consumption plan and lending (c, θ) for the lender; and a debt and repayment (φ, D) for the government such*

that

- i) (c, θ) solves the lender's problem.
- ii) (φ, D) satisfies equations (1), (2) and (3) above.
- iii) Bond markets clear: $\theta = \varphi$.
- iv) The default rate $\pi_s \in [0, 1]^S$ satisfies the following

$$D_s = (1 - \pi_s)\theta \quad (6)$$

Equation (6) states that all that is paid equals all that is received.

The market clearing condition in the asset market implies that the goods market also clears:

$$c_s = w_s + E_s^+ \quad (7)$$

Remark 2 When there exists trading in the bond market so that $\theta > 0$ and $\varphi > 0$ hold, then, item (iii), Definition 1 above and (6) imply that

$$1 - \pi_s = \frac{D_s}{\varphi} \quad (8)$$

meaning that the default rate is rationally anticipated by the lender. In other words, the payment rate $1 - \pi_s$ for the second period will equal that which is repaid D_s scaled by that which should have been paid, φ .

Note, when there is no trade in equilibrium $\theta = \varphi = 0$, π_s can no longer be determined by (8). However there could be a non-trivial equilibrium that satisfies (6) in the sense of Araújo et al. (1998). In that paper, an equilibrium is said to be non-trivial if there is asset trading or, when asset are not traded, the default rate π_s is zero. Actually, there are many ways of defining non-triviality of equilibria. Dubey et al. (2005)'s definition is one of them. Since our objective is to guarantee that in equilibrium the government is defaulting, we will leave aside the latter definition and offer the following, which allows default.

Definition 2 An equilibrium is said to be non-trivial if there is asset trading ($\varphi = \theta > 0$) and a state s such that $D_s < \varphi$. That is, there is default in equilibrium $\pi_s > 0$.²⁵

²⁵ Note Definition 2 does not exclude the possibility of $D_s = 0$ so that the default could be complete in some state of nature. And in fact it will be in the states $s \in \Gamma'$ since it is nonempty, of course.

If Γ were empty, lenders would anticipate a complete default and therefore no equilibrium since lenders would purchase nothing $\theta = 0$ thus breaking the market clearing condition (unless the government also decides to issue no debt). In the following section, we will provide a condition so that this does not happen. Accordingly, a necessary condition for the equilibrium to exist is that $\Gamma \neq \emptyset$.

5 Results

In this section we establish our main results. The first has to do with the existence of equilibrium, the second discusses its non-triviality and robustness. The section ends with some comparative static results. We begin by giving sufficient conditions which guarantee the existence of an equilibrium for $\mathcal{E}$.

Assumption 1 (Sufficient Conditions for Equilibrium). *The government's primary surplus E and the lender's utility function $u : \mathbb{R}_+^{1+S} \rightarrow \mathbb{R}$, the initial endowment $W = (w_s : s = 1, \dots, S)$ are assumed to be*

- i) $E_0 < 0$ and $\Gamma \neq \emptyset$.
- ii) Continuous, quasi concave and strictly increasing.
- iii) $W \in \mathbb{R}_{++}^{1+S}$.
- iv) $w_0 + E_0 > 0$.

Condition i) guarantees that the supply of bonds is well-defined unless the asset price is zero. The second inequality in *condition i)* rules out a complete default in all states of nature as there will be at least one state of nature where the government repays something of its debt. *Condition ii)* is standard in the GEI literature.²⁶ Strict positivity of initial endowments stated in *condition iii)* guarantees that the budget set of the lender is lower hemicontinuous which is a key component to demonstrate existence of equilibrium. *Condition iv)* guarantees that demand for bonds is well-defined and positive.²⁷

Theorem 1 (Equilibrium with Trading). *Under Assumption 1, the economy $\mathcal{E}$ has an equilibrium with trading in the bond market.*

Theorem 1 also holds when lenders are heterogeneous. Its proof directly follows from the well known generalize-game approach (Debreu, 1952, Arrow and Debreu, 1954), see Appendix B.

²⁶ Where GEI denotes general equilibrium with incomplete markets.

²⁷ We further assume that the sovereign's action do not impact the probability distribution of the different states of nature.

Remark 3 Notice that Theorem 1 only guarantees the existence of an equilibrium with trading in the bond market; it does not guarantee that the government defaults in equilibrium. In other words, we can find an equilibrium with trading and no default. That is, $D_s = E_s = \varphi, s \in \Gamma$. For that, it is enough that the government's leverage is sufficiently small. We illustrate this in the first section of Appendix C.

5.1 Non Triviality and Robustness of Equilibrium

From now on, the expected default rate will be interpreted as being *the probability of default*. Next, we give conditions for the equilibrium to be associated with default at states s for which $E_s > 0$. Since we are interested how the fiscal shocks affects the probability of default we shall impose further structure over the government's primary surplus E , and assume that the value of the second-period primary surplus suffers anticipated fiscal shocks. Finally, it is assumed that the probability distribution of the uncertainty is given by $P = (p_s : s = 1, \dots, S)$. In particular, the following holds:

Assumption 2 (Second Period Surplus).

1. The second-period primary surplus is assumed to be:²⁸

$$E_s = E_0 + A_s \tag{9}$$

where A represent some innovation or perturbation to the primary surplus E_0 , defined for each s by

$$A_s = \bar{A} \times \mathcal{I}_s \text{ with } \bar{A} > 0 \text{ its size, and with } \mathcal{I}_s = \left(1 - \frac{2(s-1)}{S-1}\right) \in [-1, 1].$$

2. The probability distribution of the uncertainty $P = (p_s : s = 1, \dots, S)$ is assumed to be strictly positive. That is, $p_s > 0, \forall s = 1, \dots, S$.

Thus $\bar{A}$ defines the support of random variable $A_s \in [-\bar{A}, +\bar{A}]$ (thus constituting absolutely equal-sized negative and positive limits, like busts and booms). $\bar{A}$ represents the maximum amplitude of exogenous shocks to the primary deficit. For instance, in the case of Argentina, one could think of this shock as a dramatic change, say, in the demand for agricultural exports (which might impact elements of the primary deficit such as tax revenues). Since A_s is decreasing with s , we have $E_{s=1} > E_{s=2} > \dots > E_{s=S}$. Moreover, whichever s is realized, $E_{s=1} = E_0 + \bar{A}$ and $E_{s=S} = E_0 - \bar{A}$, yielding the second-period deficit interval $E_0 \pm \bar{A}$.

²⁸ For simplicity we assume additive shocks. However if multiplicative, the innovation would likewise require being positive in some states, and negative in others.

Of course each of these $s \in S$ states will have an associated probability, namely p_s . Henceforth the expected value of any s -contingent variable Λ with respect to the probability $P = (p_s : s = 1, \dots, S)$ will be denoted by $\tilde{\Lambda} = \sum_s p_s \Lambda_s$. Thus, the expected value of the second-period primary surplus is

$$\tilde{E} = \sum_{s=1}^S p_s E_s = E_0 + \tilde{A} \quad (10)$$

In other words, the sum of the initial primary surplus plus the expected value of the fiscal shock across all states S . The latter can be solved as,

$$\tilde{A} = \bar{A} \left(\frac{S+1}{S-1} - \frac{2}{S-1} \sum_{s=1}^S s p_s \right) \quad (11)$$

$$-\bar{A} < \tilde{A} < \bar{A}$$

Note that if the probability of boom-like states dominate the span of S , then the expected value of the deficit will improve: $\tilde{A} > 0$ and $\tilde{E} > E_0$, and vice versa. Otherwise, if all states are equiprobable then deficits will be unchanged in expectation: $\tilde{E} = E_0$. However note that since the sovereign makes no payment in states s where $E_s < 0$, the precise probability distribution of states is not an issue. In other words, there is no constraint on P since what will matter is the positive part of E .²⁹

Remark 4 As previously mentioned, since our main interest is to know how fiscal shocks impact the default probability, the following assumption postulates a relationship between fiscal shocks and the amount of the leverage $-E_0$, under the earlier condition the economy will have a default in equilibrium³⁰. However, nothing prevents us from assuming some other condition relating the lender's income and the leverage of the sovereign. The second part of Appendix C illustrates this.

There is a *partial* default if the government makes a payment which, though a positive amount, still lies below the face value of the bond. First, partiality of default, if any, is guaranteed by Item 1 of Assumption 1. Second, Item 1 of Assumption 2 guarantees default in equilibrium as $\Gamma' \neq \emptyset$. This default is complete, though. Actually, $\pi_s = 1, \forall s \in \Gamma'$. Nonetheless, neither Assumption 1 nor Assumption 2 guarantee default at states belonging to Γ . Thus, we need something which does guarantee it. That is the role the next assumption will play.

²⁹ Being even more precise, what matters is not the strict positivity (because it could be negative) of the expected value of future primary surplus $\tilde{E}$, but rather the states in which $E_s > 0$ producing a strictly positive expected value of the repayment, D_s . That is to say, we do not rule out any probability, irrespective of how small.

³⁰ As in our earlier Remark, “default in equilibrium” refers to some state s where $E_s > 0$.

Definition 3 We say that default exists in equilibrium if there exists at least one state of nature $s \in \Gamma$ such that $\pi_s > 0$. Moreover, the equilibrium default is said to be partial if there exists at least one state of nature s such that $\pi_s < 1$.

Our next aim is to demonstrate that default exists in states s belonging to Γ , and we will do so by imposing conditions on the expected default rate with respect to the probability distribution P of states of the nature. Here it is more practical and convenient to use expected values instead of isolated, individual states of nature due to the intrinsic uncertainty of the future (see the following lemma, whose proof is straightforward).

Lemma 1 Under Assumptions 1 and 2 the following relations (items) are satisfied

1. $1 - \tilde{\pi} = \sum_{s \in \Gamma} p_s (1 - \pi_s)$
2. $1 - \sum_{s \in \Gamma} p_s \leq \tilde{\pi}$
3. $\pi_s = 1, \forall s \in \Gamma \Rightarrow \tilde{\pi} = 1$
4. $\tilde{\pi} = \sum_{s \in \Gamma'} p_s \Leftrightarrow \forall s \in \Gamma : \pi_s = 0$.

Item 2 gives the feasible expected default rates under Assumption 2. Item 3 guarantees that the equilibrium default is partial. Notice that Item 4 says that we could have an strictly positive expected default rate $\tilde{\pi}$ without having default in equilibrium in the sense of Definition 3. However, Item 3 will give us hints to show that default exists in equilibrium. More precisely we have the following:

Assumption 3 (Initial Primary Surplus). The primary initial surplus E_0 is assumed to be characterized by,³¹

$$-E_0 \left(\frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s} \right) < \bar{A} < -2E_0 \left(\frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s} \right) \quad (13)$$

Note, the term in parentheses is strictly above 1. Defining $\alpha = -\frac{\bar{A}}{E_0} > 1$, (13) is equivalent to

$$\frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s} < \alpha < 2 \left(\frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s} \right) \quad (14)$$

³¹ In equilibrium we have that (1) and (3) hold. Combing (1) and (3) and using the definition $1 - \tilde{\pi}$ we have that

$$-E_0 \left(\frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s} \right) \leq \bar{A} \leq \frac{-E_0(1+r)}{\sum_{s \in \Gamma} p_s \mathcal{I}_s} \quad (12)$$

Thus (12) is a necessary condition for the existence of equilibrium with $\varphi = \theta > 0$ But (13) implies (12) so that (13) is a sufficient condition for default to be partial, i.e., $1 - \sum_{s \in \Gamma} p_s < \tilde{\pi} < 1$. In terms of intuition, (12) or (13) says that fiscal disturbances must be strictly positive and limited at the top for default to be partial.

This term α , which we shall call the *disturbance*, captures how small or large the initial primary surplus E_0 is in relation to the maximum size $\bar{A}$ of the fiscal innovation suffered by that initial surplus; in other words, how much of the initial deficit may at maximum be “disturbed” in the second period.

In what follows we shall prove that, under Assumption 3, the expected default rate, $\tilde{\pi}$, satisfies: $1 - \sum_{s \in \Gamma} p_s < \tilde{\pi} < 1$. This fact, together with Lemma 1, implies that default exists in equilibrium and is partial according to Definition 3. More precisely, we have the following proposition, and the following key equation:

Proposition 1 *Assumptions 1 and 2 imply that*

$$1 - \pi_s = \frac{E_0 + \bar{A}\mathcal{I}_s}{-(1+r)E_0}, s \in \Gamma \quad (15)$$

Under the same hypotheses of Lemma 1 together with Assumption 3, the expected rate of default in equilibrium $\tilde{\pi}$ satisfies

$$1 - \tilde{\pi} = \frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{1 + r}. \quad (16)$$

In addition there exists default in equilibrium, and it is partial.

Proof: Appendix A.

Equation (16) of Proposition 1 represents the *fundamental equation* of the paper. It declares that the probability of default depends on the interest rate on public debt r ; on the fiscal disturbance $\alpha > 0$; on the range of uncertainty given by S ; and the probabilities $p_s > 0$ of the states where $E_s > 0$. Thus for expositional purposes we can graph default outcomes in $(1 + r, \alpha)$ or $(1 + r, p_s)$ space (as we do later). This also allows us, akin to a Slutsky decomposition, to movements within and across a default schedule.

In order to identify the determinants of the endogenous variables (such as the default rate and the cost of the public debt) and analyze their inter-relations, we can rewrite (16) as

$$(1 + r)(1 - \tilde{\pi}) = \sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1) \quad (17)$$

In the language of Dubey et al. (2005), the left side of (17) represents the lender’s receipts per unit of bond purchased which, in equilibrium, are nothing other than the rate of payment that the government makes per “each unit of bond” issued.

To state that the government does not default³² means $\pi_s = 0, \forall s \in \Gamma$ which is equivalent to $\tilde{\pi} = \sum_{s \in \Gamma'} p_s$ by Item 4 of Lemma 1. Thus, $\tilde{\pi} \neq 1$ as $\sum_{s \in \Gamma} p_s \neq 0$ by Item 2 of Assumption 2. Hence, the default-free gross rate of return, which we shall denote by $1 + i$, is computed from (17) after substituting $\tilde{\pi} = \sum_{s \in \Gamma'} p_s$. Thus,

$$1 + i = \frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{\sum_{s \in \Gamma} p_s} \quad (18)$$

Remark 5 Condition (16) says that the probability of default and the interest rate of the public debt in times of crises are determined by a unique equation.³³ By “crises” we mean a situation in which $1 - \sum_{s \in \Gamma} p_s < \tilde{\pi} < 1$. A sufficient condition for “crises” to exist is given either by (13) or (14), which are expressed in fiscal terms. However, as shown in the second part of Appendix C, sufficient conditions in terms of the lender’s preferences are given to guarantee $1 - \sum_{s \in \Gamma} p_s < \tilde{\pi} < 1$. For example, for a risk-neutral lender, a sufficient condition for default to be partial is that the lender’s impatience rate, at default state, belongs to $(0, 1)$ by (14). See equation (E.15) of Appendix E. In equilibrium these conditions should naturally be interrelated.

Note, that in the limiting case, $\tilde{\pi} = \sum_{s \in \Gamma'} p_s \Leftrightarrow \pi_s = 0, \forall s \in \Gamma$ (i.e., no default), the return on the sovereign debt, $1 + i$, is determined by (18) whereas in the limit case, $\tilde{\pi} = 1$, (complete default and thus worst-case scenario) these variables are indeterminate, as shown in the following corollary.

Corollary 1 Under the first two assumptions of Proposition 1, the following is satisfied

$$\tilde{\pi} = 1 \text{ if } \alpha = \frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s}$$

and $1 + r$ is indeterminate.

Proof: Appendix A.

5.2 Impacts of Fiscal Disturbances

Recall that α captures how small or large the initial primary surplus E_0 is in relation to the maximum size $\bar{A}$ of the fiscal disturbance suffered by E_0 . Accordingly, our next result shows how both the probability

³² Notice that “no default” does not mean $\tilde{\pi} = 0$ but rather $\tilde{\pi} = \sum_{s \in \Gamma'} p_s$.

³³ As stated in Geanakoplos (2010), typically in private borrowing, theory either ignores the possibility of default (and thus the need for collateral) or fixes the leverage as a constant (as in Mendoza, 2010) allowing a single supply-equals-demand equation to predict the interest rate. But, since in the Sovereign indebtedness there is no collateral, then lenders anticipate the default rate so that in crises (partial default) the demand for public debt will depend both on interest rate and the default rate. So, the equilibrium of supply and demand will determine both the equilibrium default and the interest rate.

of default and the cost of the public debt are³⁴ affected by α .

Proposition 2 *In a partial-default regime, $\tilde{\pi} \in (1 - \sum_{s \in \Gamma} p_s, 1)$, the following holds*

$$\frac{\partial \tilde{\pi}}{\partial \alpha} < 0 \text{ and } \frac{\partial(1+r)}{\partial \alpha} > 0. \quad (19)$$

Proof: Appendix A.

Proposition 2 states that the impact of fiscal disturbance α on $\tilde{\pi}$ and r is of contrasting signs. The intuition is as follows. First, keeping exogenous $\bar{A}$ constant, the smaller $-E_0$ is, the less is the government leverage. By leverage we mean the amount of debt used to finance assets: recall in equation (1), $-E_0 = q\varphi$ is the value of the public debt: the smaller $-E_0$ is, the less is government leverage (indebtedness). Therefore, the probability of default would tend to decrease which would imply an increase of the demand of the public debt making q falls and r increases. Second, keeping E_0 constant, the stronger is the fiscal shock $\bar{A}$, the greater the expected payment $\tilde{D} = \tilde{E}^+$ is by (2) and (10). Thus, the probability of default would tend to decrease so that by the same argument above its price q would fall (and r would increase) due to lenders would purchase more public bonds.

5.3 Impacts on Economic Performance

Just as α influences both the probability of default and the cost of the public debt, the probability distribution, P , can also be shown to impact these variables, specifically so the probability of the states s belonging to Γ , p_s . Due to the fact that the primary surplus, in states belonging to Γ , exceeds 0, the economy is said to be in a good level of economic performance.

Thus, under the same conditions of Proposition 2 and using the same argument of its proof, we conclude:

$$\frac{\partial \tilde{\pi}}{\partial p_s} = -\frac{\alpha \mathcal{I}_s - 1}{1+r} < 0 \text{ and } \frac{\partial(1+r)}{\partial p_s} = \frac{\alpha \mathcal{I}_s - 1}{1-\tilde{\pi}} > 0. \quad (20)$$

If we interpret the probability p_s as the economy's level of performance, these terms demonstrate that the impact of an economic improvement on $\tilde{\pi}$ and r will be negative and positive, respectively (conditional on α and $s \in \Gamma$). This is due to the fact that $\alpha \mathcal{I}_s - 1 > 0$ (as the primary surplus, in states belonging to Γ , exceed 0).

³⁴ Actually these endogenous variables are determined by a unique equation so that it would be sufficient to study the impact of α on one of these variables in question.

The intuition behind the first relation is the following: if it is expected that the probability of a good economic performance will increase p_s , the economy is expected to have a higher future surplus leading to a higher repayment which would straightforwardly mean that the expected default rate falls. This implies that lender would demand more public debt by making its price q fall, with the gross rate increasing.

Remark 6 *First, we know that whichever we increase by α or p_s the expected default rate will increase by (19) and (20) respectively. In either case, this would cause an increase in θ and therefore in φ as well since the economy is equilibrium. Second, given that $q\varphi = -E_0$ also holds in equilibrium; then keeping E_0 unchanged, an increase in φ would imply a fall in q and therefore an increasing in $1 + r$. Finally, if α increased due to a decrease in $-E_0$, then an increase in φ would provoke a increase of q even more pronounced.*

We have just seen that in a default partial regime an increase in p_s , independent of the size of the future surplus E_s , with $s \in \Gamma$, causes an increase in the gross interest rate $1 + r$. This is true because the expected default rate and the gross interest rate are determined by unique equation, namely (16). However, in non-default regime, this is not the case.

Corollary 2 *Suppose that $S > 3$. Then, the impact of p_s on the default-free gross interest rate $1 + i$ is*

1. *positive if $s = 1$,*
2. *negative if $s = \lfloor S/2 \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to the number x , and*
3. *ambiguous if $1 < s < \lfloor S/2 \rfloor$.*

Proof: Appendix A.

5.4 How Fiscal Disturbances and Booms Modify the Cost of Public Debt

Proposition 1 and Corollary 1 can be interpreted geometrically. To do so, consider the simplified case where there are only two states of nature, $S = 2$ and two associated probabilities $p_1 = p$ and $p_2 = 1 - p$.³⁵ Under these conditions, the probability of default $\tilde{\pi}$ as defined³⁶ in (16), can be viewed as a function of $(\alpha, 1 + r)$. However, to include the limit cases of no default and complete default we shall consider the

³⁵ Appendix G considers the case of $S > 2$ in the context of changes in the partial default cone.

³⁶ After solving for $1 + r$ since $\tilde{\pi} \neq 1$.

locus defined by (17) for this special case. Thus, for a fixed probability $0 < p < 1$ of a boom, we have that for each expected default level $\tilde{\pi} = \gamma \in [1 - p, 1]$, (17) becomes

$$(1 - \gamma)(1 + r) = p(\alpha - 1), \quad \underbrace{\frac{1 - \gamma}{p} + 1}_{\alpha_{(1)}} \leq \alpha \leq \underbrace{\frac{2(1 - \gamma)}{p} + 1}_{\alpha_{(2)}} \quad (21)$$

where $[\alpha_{(1)}, \alpha_{(2)}]$ defines the interval containing all α such that $1 + r \in [1, 2]$.³⁷

For the case $\gamma = 1 - p$, equation (21) describes the cost of the default-risk public debt, denoted by $1 + i$, as a function of the fiscal disturbance α . More precisely, using Item 1 of Lemma 1 we have

$$\text{No Default : } 1 + i = \alpha - 1, \quad 2 \leq \alpha \leq 3 \quad (21')$$

after substituting $\gamma = 1 - p$ both in $\alpha_{(1)}$ and $\alpha_{(2)}$ respectively.³⁸

For $\gamma = 1$, (21) becomes,

$$\text{Complete Default : } \alpha = 1, \quad \text{with } 1 + r \text{ undetermined} \quad (21'')$$

Hence, the cost of the public debt for each default level γ is

$$\text{Partial Default: } 1 + r = \frac{p}{1 - \gamma}(\alpha - 1), \quad \alpha_{(1)} \leq \alpha \leq \alpha_{(2)} \quad (22)$$

Figure 5 illustrates. It shows all curves $1 + r$ as a function of α whose slope $p/(1 - \gamma)$ varies with α for a fixed p with $1 - p \leq \gamma < 1$. If we rotate the level curve, associated to level $\gamma = 1 - p$, anti-clockwise (as indicated by the black arrow) until we reach the boundary $\gamma = 1$, we would have swept through all the intermediate γ levels. Note, that we shall use these blue-red-green combinations in all subsequent figures to capture no default, complete default and partial default outcomes, respectively.

As the default probability tends to unity in the limit, we have the associated vertical line $\alpha = 1$ (i.e., where $1 + r$ is indeterminate). Moreover, and quite naturally, for a given fiscal disturbance, say $\alpha = \alpha_{(1)}$, and a given boom probability, it can be seen that the (interest) cost of debt increases (implying a leftward rotation of the green line and the arrow) as the sovereign becomes more likely to default. Note, comparing the partial and no-default cases, respectively, that $\alpha_{(1)} < 2$ and $\alpha_{(2)} < 3$. This is consistent

³⁷ 1, 2 is a presumed range for the gross interest rate.

³⁸ In other words, $1 + i$ is the cost of the public debt $1 + r$ when $\gamma = 1 - p$: when the sovereign does not default in the state where $E_1 > 0$. That is, $\pi_1 = 0$ by Item 1 of Lemma 1.

with the understanding that a defaulting sovereign will tend to be more leveraged than the one not defaulting. In other words, for a given $\bar{A}$, E_0 will be absolutely greater for the defaulter than the non defaulter.

Thus partial default lies in the interior of the cone³⁹ of outcomes. Given that much of the literature has considered sovereign default as a binary outcome, it is worth pointing out that partial default can be shown to be not only an equilibrium outcome but a robust one (in a topological sense). The proof and analysis of the robustness of the default limit case and the no-default limit case is carried out in Appendix D and relies upon a topological analysis of key models conditions.

This (stability) precisely accords with many historical experiences: under extreme fiscal stress, complete debt repudiation or zero default outcomes are considerably less common than periods of partial default and, in practice, re-negotiation (for an analysis of the historical frequency of default, see Tomz and Wright, 2013).

6 Endogenizing Fiscal Shocks and Balances: The Role of Habits

Let us recap: we considered the case of an innovation to the primary surplus which provides an upper and lower boundary for the expected second period deficit, $\tilde{E}$. Depending on the probability distribution of the states of nature, the realized interval for E_s may or may not reach those extremes. To guarantee trading at least one innovation must yield a positive primary surplus, implying the lack of a complete default.

The treatment of default that we have considered so far, has the drawback that sovereign default essentially reflects initial conditions and shocks. This was a useful starting point since it allowed us to consider how, in contrast to the binary case, partial default can be an equilibrium outcome. However, it is not fully satisfactory since it sidesteps the sovereign's own possible role in default episodes.

Accordingly we supplement this analysis and consider the case where it is not purely exogenous innovations that drive default events, but the sovereign's own fiscal efforts. By "fiscal efforts" we mean any conduct that the government adopts to modify either public expenditure or taxes to manage fiscal obligations; different effort levels will lead to different primary surplus outcomes. We assume that these fiscal efforts will, in turn, be guided by *habit persistence* with respect to the reference levels of shocks,

³⁹ Strictly speaking, what we have is a truncated cone whose bases belongs to the horizontal lines representing the minimum and maximum gross rate of return respectively. However, henceforth we will not refer to them explicitly any since the critical situations are in those of no default and complete default cases, represented by blue and red lines, respectively.

implying that actions may be anchored by past fiscal efforts and fiscal histories. This is consistent with Section 2, which discussed persistence in deficit profiles and the empirical relevance of fiscal rigidities and biases.

6.1 Fiscal Efforts and Habit Formation

Let us assume that there is an one-to-one correspondence between the fiscal effort levels and the maximum deviations suffered by second-period primary surplus plan $(E_1, \dots, E_S)$ due to the innovations to the first-period primary surplus E_0 . We denote such deviations by η .

Remark 7 *Under the earlier assumption, each time we refer to fiscal effort levels we will be referring to deviations η themselves. For the sake of clarity, we stress for example that the exogenous deviation $\bar{A}$ (called the fiscal shock, in the previous section, which, together with E_0 , determined fiscal disturbances α) was caused by a fiscal effort, say $\bar{\eta}$. That is now $\bar{\eta} \equiv \bar{A}$.*

Knowing that η and $\bar{A}$ have the same nature, we postulate that the government forms fiscal habits leading to the following assumption.

Assumption 4 (Fiscal Memory). *Under endogenous fiscal effort, we assume that the maximum amplitude of the shocks, i.e. the fiscal deviations, will now depend on fiscal efforts. To that end assume fiscal interventions follow a habit-formation process. The level of fiscal interventions in the current period, η_1 , depend on the past intervention level η_0 and levels of $\bar{A}$ which determines the maximum amplitude of $\tilde{A}$:*

$$\eta_1 = \rho\eta_0 + (1 - \rho)\bar{A} \quad (23)$$

Parameter $\rho \in (0, 1]$ indexes the persistence or “memory” in the habit-formation process. The closer ρ is to unity, the more current fiscal interventions are anchored to past ones. Notice that (23) is a quite reasonable simplification of the habit persistence (or habit formation) commonly used in various contexts (e.g., Levine et al., 2012). Moreover including $\bar{A}$ as a reference point facilitates comparisons with the earlier exogenous case.

Furthermore, re-writing (23) as,

$$\eta_1 = \rho(\eta_0 - \bar{A}) + \bar{A} \quad (24)$$

makes clear that fiscal interventions are not an unambiguous process: they can increase or decrease

with ρ depending on $\text{sgn}(\eta_0 - \bar{A})$.⁴⁰ If we think of $\eta_0 > 0$ as a past fiscal intervention level associated with the current primary surplus E_0 , then the fiscal intervention level η_1 that will originate with the future primary surplus E_s is going to depend on the comparison between the inherited stock of habits η_0 and the reference level $\bar{A}$. For example if the past outcome was even more extreme than the maximum amplitude of the reference shock, then current fiscal behavior will inherit some of that compact. The alternative, we dub ‘good’ habits: $\bar{A} > \eta_0$ and $\eta_1 > \eta_0$.

These scenarios give a flavor of how fiscal habits might affect default outcomes. The deficit is, as before, still bound by $E_0 + \bar{A}$ but it will have theoretically fatter tails depending on the ρ value. In what follows we show this fact in several steps ordered in a convenient way. We begin by making fiscal efforts – and as a consequence also the disturbances – endogenous .

6.2 Endogenous Fiscal Effort under Bad Habits

If, following Remark 7, we consider these deficit deviations as being underpinned by habit rule (23), then the deviations become a function of the memory process, and the intervention levels. Accordingly, there is an endogenous fiscal effort as represented by,

$$\bar{A}(\rho) = \frac{\eta_1 - \rho\eta_0}{1 - \rho} \quad (25)$$

Similarly, we can define endogenous fiscal disturbances as,

$$\alpha(\rho) = -\frac{\bar{A}(\rho)}{E_0} \equiv \frac{\eta_1 - \rho\eta_0}{1 - \rho} \frac{1}{-E_0} \quad (26)$$

We write these as $\bar{A}(\rho)$ and $\alpha(\rho)$, reflecting that they are now affected by the sovereign’s fiscal habits. For comparison purposes, it is useful to note that there is a level of ρ for which $\bar{A}(\rho) = \bar{A}$ and thus $\alpha(\rho) = \alpha$, denoted by ρ_x :

$$\rho_x = \frac{\eta_1 - \bar{A}}{\eta_0 - \bar{A}} < 1, \quad (27)$$

Thus, when $\rho = \rho_x$ the endogenous fiscal effort boundary coincides with that of the exogenous boundary (note that this boundary is associated to a below-unitary memory). The variations that both $\bar{A}(\rho)$ and $\alpha(\rho)$ experience due to changes in memory ρ are going to depend on which particular habits the

⁴⁰ The sign(um) function of a real number h , is defined by: $\text{sgn}(h) = -1$ or 1 depending on whether h is a negative or positive real number. Moreover $\text{sgn}(0) = 0$.

government forms, as the below lemma shows:

Lemma 2 *The following holds:*

$$\text{sgn}\left(\frac{\partial \bar{A}(\rho)}{\partial \rho}\right) = \text{sgn}(\eta_1 - \eta_0) \quad (28)$$

Since $\alpha(\rho)$ is determined by $\bar{A}(\rho)$ it follows that $\text{sgn}(\partial \alpha / \partial \rho) = \text{sgn}(\eta_1 - \eta_0)$. More specifically, if a government forms bad habits

$$\eta_1 \leq \eta_0 \Leftrightarrow \bar{A} \leq \eta_0 \quad (29)$$

then the following holds:

$$\eta_0 > \bar{A} \Leftrightarrow \frac{\partial \bar{A}(\rho)}{\partial \rho} = \frac{\eta_1 - \eta_0}{(1 - \rho)^2} < 0. \quad (30)$$

In other words, if the fiscal authority keeps the policy regime rigidly close to that which brought about E_0 , then endogenous fiscal efforts will weaken. Moreover, since $\delta^2 \bar{A}(\rho) / \delta \rho^2$ is also negative, then fiscal effort under bad habits will weaken further in the memory parameter. To echo our earlier example, if Argentina is impacted by an exogenous reduction in exports, this would mean the government would lose control of tax revenue and would have to re-establish management of the primary deficit through expenditures. However if they showed no such flexibility, the deviations of the expected second-period primary deficit from the first would widen. The opposite conclusions follow under good habits. The question that follows is whether this simple additional rule can shed light on our previous sovereign default model. Before then we summarize.

Summary Given the fiscal efforts η_0 and $\bar{\eta} \equiv \bar{A}$, first the fiscal efforts η_1 are determined via (23). Second, the endogenous fiscal effort $\bar{A}(\rho)$ is determined via the fiscal rule (25). Third, the endogenous future primary surplus $E_s = E_0 + \bar{A}(\rho)\mathcal{I}_s$ is determined, which will impact either the expected default rate or the cost of the public debt which are determined by Proposition 1. Fourth, if $\rho = \rho_x$ then we recover the exogenous future primary surplus associated to Assumption 2.

Finally, if $\rho = \rho_0 = (\eta_1 + E_0) / (\eta_0 + E_0)$, the endogenous fiscal effort $\bar{A}(\rho)$ at ρ_0 is E_0 , which is the same as stating the endogenous fiscal disturbance is unitary at ρ_0 . In others words, as will be showed later, at ρ_0 the situation will be one of complete default.

6.3 Feasible Memory Parameters

In what follows we can be more specific and pin down which memory parameters are associated to the three debt regimes: complete, partial and no default. We will refer to these parameters throughout this section as *feasible parameters*.

Lemma 3 *Suppose the government is in a bad-habit regime so that (29) holds. Then, there is a partial default if*

$$\frac{[\eta_1 + 2E_0(P)]^+}{\eta_0 + 2E_0(P)} < \rho < \frac{\eta_1 + E_0(P)}{\eta_0 + E_0(P)} \quad (31)$$

where $E_0(P) = E_0\left(\frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s}\right)$.

Proof: Appendix A.

Let us denote the ratios on the left and right side of (31), respectively, as ρ_{min} and ρ_{max} . Then the following must be satisfied:⁴¹

$$\rho_{min} < \rho_x < \rho_{max} < \frac{\eta_1}{\eta_0} < 1. \quad (32)$$

Moreover, we also have

$$\begin{aligned} \bar{A}(\rho_{max}) &< \bar{A}(\rho) < \bar{A}(\rho_{min}) \Leftrightarrow \\ \alpha(\rho_{max}) &< \alpha(\rho) < \alpha(\rho_{min}) \end{aligned}$$

since the function $\bar{A}(\cdot)$ is decreasing under bad fiscal habits. In other words, under bad fiscal habits the maximum width of the disturbance is decreasing in the memory parameter. This fact will play an important role on the impact that fiscal habits have on the public debt cost and therefore on the default risk premium, an important concept to be addressed later.

6.3.1 A Precise Boundary for $\bar{A}$

Clearly $\bar{A}(f(x)) = x$. Thus, substituting $f(x)$ by ρ_{min} and ρ_{max} , respectively, one can define a boundary for $\bar{A}$ such that $-E_0(P) < \bar{A}(\rho) < -2E_0(P)$ provided that $\eta_1 > -2E_0(P)$. Otherwise $-E_0(P) < \bar{A}(\rho) <$

⁴¹ The inequalities in (32) follow from the fact that under bad habits, $(\eta_0 > \eta_1)$ the function $f(x) = \frac{\eta_1 - x}{\eta_0 - x}$ is strictly decreasing for every $x > 0$.

η_1 . This means that if the current fiscal effort η_1 is below $-2E_0(P)$, then the values of the endogenous shock fiscal (or disturbances) become more concentrated to the left becoming less volatile.

6.4 Impact of Fiscal habits on the Default Rate

We know from (32) that the exogenous memory ρ_x lies in the interior of the interval for the feasible “memory” parameter. Thus, it could transpire that $\bar{A}(\rho)$ is above or below $\bar{A}$, depending on whether ρ is below or above ρ_x , respectively. Since $\bar{A}(\cdot)$ is decreasing, it does not matter what happens on the left end of the feasible interval of ρ ; what we shall observe is that under bad habits, ρ has an upper bound below 1. This implies that if the government insists on implementing the maximum memory level ρ_{max} , there will be a complete expected default: the sovereign does not even need to stay completely in past habits (i.e., $\rho = 1$) to experience a complete default (i.e. $\tilde{\pi} = 1$). This follows from (17) with α endogenous. Therefore, we have that $\tilde{\pi} = 1$ if $\alpha(\rho) = \frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s}$ wherein $1 + r$ is indeterminate.

On the other hand, the no-default case is reached by substituting $\tilde{\pi} = 1 - \sum_{s \in \Gamma} p_s$ into (17) with α endogenous. That is

$$\text{No Default: } (1 + i)(\alpha) = \frac{\sum_{s \in \Gamma} p_s \left(\alpha(\rho) \mathcal{I}_s - 1 \right)}{\sum_{s \in \Gamma} p_s} \quad (33)$$

where $(1 + i)(\alpha)$ means $1 + i$ is a function of α .

Finally, for the partial-default regime $\gamma \in (1 - \sum_{s \in \Gamma} p_s, 1)$, the cost of the debt is

$$\text{Partial Default: } 1 + r = \frac{\sum_{s \in \Gamma} p_s \left(\alpha(\rho) \mathcal{I}_s - 1 \right)}{1 - \tilde{\pi}}. \quad (34)$$

We know that defaults are accompanied by interest rate spikes (Arellano, 2008). It is therefore of interest to now if incorporating habits into our sovereign debt framework can speak to this fact. The following analysis is carried out under bad habits. Let us note that an important consequence of (34) is that if the government forms bad habits $\eta_0 > \eta_1$, the expected repayment $\tilde{D}$ will decrease as ρ increases. This is because in equilibrium $\tilde{D} = \tilde{E}^+$, and from the fact that $\tilde{E}^+$ depends on α . Thus, in the presence of bad habits, the expected default rate increases as ρ increases, provided the interest rate r to be kept fixed, as the following corollary states.

Clearly, any change in the fiscal disturbance α makes the default curve⁴² moves away from, or closer

⁴² Defined in terms of gross interest rate and payment rate and given in (34).

to, the origin depending on whether α increases or decreases⁴³ respectively. Notice that, in general, the impact that α has on both the expected default rate and on gross interest rate is ambiguous and obeys the following relation, which is obtained after differentiating (34) with respect to memory level.

$$-(1+r)\frac{d\tilde{\pi}}{d\rho} + (1-\tilde{\pi})\frac{d(1+r)}{d\rho} = \sum_{s \in \Gamma} p_s \alpha'(\rho) \mathcal{I}_s \quad (35)$$

However, if we hold constant any of the endogenous variables, either $1+r$ or $\tilde{\pi}$, the impact ρ has on the earlier variables depends whether on the fiscal habits are bad or not, as shown in the following corollary.

Corollary 3

$$\text{sgn}\left(\frac{\partial \tilde{\pi}}{\partial \rho}\right) = \text{sgn}(\eta_0 - \eta_1) \quad \text{sgn}\left(\frac{\partial(1+r)}{\partial \rho}\right) = \text{sgn}(\eta_1 - \eta_0) \quad (36)$$

Proof: Appendix A.

Corollary 3 states that the impact the habit-formation process has on the expected default rate is not unique, but will depend on the distance $\eta_1 - \eta_0$. Intuitively, if the government forms bad fiscal habits, $\eta_0 > \eta_1$, according to (29) we have that $\bar{A} < \eta_0$. This indicates that the η_0 component is substantive, then increasing the speed of adjustment ρ must yield an increase in the probability of default. The parameter ρ will naturally not have an impact if the lender believes that $\bar{A}$ is fixed (because it is the lender who anticipates the default).⁴⁴ As before, $\tilde{\pi}$ and $1+r$ move in opposite directions: thus any change in ρ than reduces $\tilde{\pi}$, will increase $1+r$.

6.4.1 Impact of Fiscal habits on Economic Performance

Finally, in the presence of α depending on a fixed ρ , both the impact and economic intuition that the probability p_s of states s belonging to Γ have on $\tilde{\pi}$ and $1+r$ are the same qualitative results as those of the earlier exogenous fiscal disturbances case:

$$\frac{\partial \tilde{\pi}}{\partial p_s} = -\frac{\alpha(\rho)\mathcal{I}_s - 1}{1+r} < 0 \quad \text{and} \quad \frac{\partial(1+r)}{\partial p_s} = \frac{\alpha(\rho)\mathcal{I}_s - 1}{1-\tilde{\pi}} > 0. \quad (37)$$

⁴³ For example, under bad fiscal habits $\eta_0 > \eta_1$, α decreases as shown in (30).

⁴⁴ In other words, if the endogenous fiscal effort $\bar{A}(\rho)$ equals exogenous $\bar{A}$, then as lenders anticipate the fiscal rule, they should expect that it does not impact the probability of default.

Note that under bad habits $\partial^2 \tilde{\pi} / (\partial \rho \partial p_s) > 0$ the impact of expected booms on default is tempered by the presence of habits. In other words, a bad-habit sovereign requires a stronger-than-normal boom to reduce default probabilities.

6.5 How Habits Modify the Cost of Public Debt

To illustrate geometrically the results we specialize (17) with α being endogenous for the simplifying $S = 2$ case, and with p being the probability of a boom. This geometric analysis will be carried out in the plane $(1 + r, \rho)|_{p_s}$ under bad fiscal habits. Note that the case showed in Figures 6 is the corresponding endogenous version of Figures 5 (written in terms of ρ instead of α since we know under habits that $\alpha(\rho)$).

6.5.1 Influence of Fiscal Habits

We begin by analyzing how habits modify default outcomes. For that, we consider a non-linear version of (21):

$$(1 + r)(1 - \gamma) = p(\alpha(\rho) - 1), \quad \rho_{(1)} \leq \rho \leq \rho_{(2)} \quad (38)$$

where $1 - \gamma$ can be interpreted as being all payment rates with $\gamma \in [1 - p, 1]$. The values $\rho_{(1)} = \frac{\eta_1 + E_0 \left(\frac{2(1-\gamma)}{p} + 1 \right)}{\eta_0 + E_0 \left(\frac{2(1-\gamma)}{p} + 1 \right)}$, and $\rho_{(2)} = \frac{\eta_1 + E_0 \left(\frac{1-\gamma}{p} + 1 \right)}{\eta_0 + E_0 \left(\frac{1-\gamma}{p} + 1 \right)}$ determine all ρ such that $1 + r \in [1, 2]$.

The no-default case is obtained from (38) implementing $\gamma = 1 - p$. Thus, we have

$$\text{No Default: } (1 + i)(\rho) = \alpha(\rho) - 1, \quad (39)$$

with $\frac{\eta_1 + 3E_0}{\eta_0 + 3E_0} \leq \rho \leq \frac{\eta_1 + 2E_0}{\eta_0 + 2E_0}$.

For the other limit case $\gamma = 1$, we have that from (38)

$$\text{Complete Default: } \alpha(\rho) = 1$$

so that the memory parameter associated to a zero gross rate is given by $\rho_{(0)} = \frac{\eta_1 + E_0}{\eta_0 + E_0} < 1$, with $1 + r$ being undetermined. The intermediate partial default cases given by $\gamma \in (1 - p, 1)$ are obtained from (38) by solving for $1 + r$:

$$\text{Partial Default: } 1 + r = \frac{p(\alpha(\rho) - 1)}{1 - \gamma}, \quad \rho_{(1)} \leq \rho \leq \rho_{(2)} \quad (40)$$

Figure 6 shows all curves $1 + r$ as function of ρ with fixed p and γ varying in the interval $[1 - p, 1]$.

Remark 8 *It is useful to note that all curves start from $\rho_{(0)}$. Moreover, for the blue no-default curve, $\gamma = 1 - p$, to intercept the horizontal bands $[1, 2]$ we require in (39) $\eta_1 \geq -3E_0$. In Figure 6, the case $\eta_1 = -3E_0$ is shown. Finally, as long as $\gamma \rightarrow 1$ we observe that the green curves tends to the red curve while its interval $[\rho_{(1)}, \rho_{(2)}]$ collapses to the point $\{\rho_{(0)}\}$ which represents the limit case, namely that of a complete default $\gamma = 1$.*

Now, to gauge the impacts caused by the “memory” levels on the probability of default and the gross rate of interest, we use Corollary 3 for the case $\eta_0 > \eta_1$. Thus,

$$\frac{\partial \tilde{\pi}}{\partial \rho} > 0 \quad \text{and} \quad \frac{\partial(1 + r)}{\partial \rho} < 0. \quad (41)$$

These are the same conditions as under Corollary 2 but now specifically shown for bad habits.

Intuitively, increasing the “memory” level, the fiscal effort defined by (25) decreases due to the bad fiscal habits. Thus, the expected repayment, $\tilde{E}^+$, would fall, implying a increase in the expected rate of default (which is interpreting as being a proxy for the probability of default). If this happens the lender would demand less public bonds so that its price q would increase and $1 + r$ would decrease. Geometrically, this is shown by the negatively sloped green curves in Figure 6.

7 Sovereign Default Risk Premium, r^{rp}

So far we have only considered one interest rate in the economy. In this and the following section, we shall deal with two closely related concepts in a sovereign-default environment:

- **r^{rp} : Sovereign Default Risk Premium:** i.e., the additional amount paid to compensate a lender for assuming sovereign default risk.⁴⁵
- **$\mathcal{C}$: Sovereign Cost of Default:** i.e., the cost the sovereign incurs when it repudiates its debt (to be discussed in section 8). The costs of default for a sovereign, is problematic due to the lack of tangible collateral backing the public debt, see Wright (2013). Accordingly, we do not address this issue with any collateral requirement or either reputation, but offer a formulation for the cost of sovereign default in terms of its leverage (or indebtedness).

⁴⁵ In a careful, historical analysis, Meyer et al. (2019) maintain that the returns to sovereign bonds have been sufficient to compensate for risk.

In what follows we see how both r^{rp} and $\mathcal{C}$ are affected by the fiscal disturbances and, in turn, by habits. Finally, we delineate a relation between the two terms.

7.1 Sovereign Default Risk Premium

Here we shall determine which factors impact the sovereign default risk premium. Naturally, to proceed, we first need to establish which is the risk-free rate of return to be used. The rate of return which is more convenient to be used is the *default-free* rate of return $1 + i$, defined in (18), instead of using the default-free gross interest rate R which is a general measure used in the economy to discount any future cash flow and future income stream.⁴⁶ Under standard assumptions, such as risk neutrality, we can without loss of generality consider $1 + i$ and R as interchangeable. Indeed, Appendix E shows an example of an economy with a risk-neutral lender in which the three rates R , $1 + i$ and $1 + r$ are compared.

Our main objective in this section is to ascertain how α and ρ affect the default risk premium, defined as,

$$r^{pr} \equiv (1 + r) - (1 + i) \quad (42)$$

$$r^{pr} = \left(\frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{1 - \tilde{\pi}} \right) - \left(\frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{\sum_{s \in \Gamma} p_s} \right) \quad (43)$$

whenever $1 - \tilde{\pi} \neq \sum_{s \in \Gamma} p_s$. The first term on the rhs in (42) can be obtained from (16) and the second from (18), yielding (43). Thus to evaluate any impact of $\alpha, \tilde{\pi}, p_s$ etc on r^{pr} implies impacts on both $1 + i$ and $1 + r$ components.

7.2 How Fiscal Disturbances Modify the Default Risk Premium

We begin by analyzing the exogenous case (i.e., no fiscal habits). Clearly, r^{pr} depends on the endogenous variable $\tilde{\pi}$, as well as on the exogenous variables α and on p_s . Holding p_s fixed, r^{pr} becomes a function of two variables: $(\alpha, \tilde{\pi})$. Computing the total differential of the default risk premium after the marginal

⁴⁶ See LeRoy and Werner (2000, p.61) for a definition.

changes in those determinants, we have:

$$\begin{aligned}
dr^{pr} &= \frac{\partial r^{pr}}{\partial \alpha} d\alpha + \frac{\partial r^{pr}}{\partial \tilde{\pi}} d\tilde{\pi} \\
&= \underbrace{\left(\frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{1 - \tilde{\pi}} - \frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{\sum_{s \in \Gamma} p_s} \right)}_{\text{Term 1}} d\alpha + \underbrace{\frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{(1 - \tilde{\pi})^2}}_{\text{Term 2}} d\tilde{\pi}
\end{aligned} \tag{44}$$

where we shall treat the two under-braced terms separately. ‘

To see specifically how the α component impacts the premium we make it only depend on α . From (44) we have,

$$\begin{aligned}
(r^{pr})'(\alpha) &= \frac{\partial r^{pr}}{\partial \alpha} + \frac{\partial r^{pr}}{\partial \tilde{\pi}} \times \frac{\partial \tilde{\pi}}{\partial \alpha} \\
&= \left(\frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{1 - \tilde{\pi}} - \frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{\sum_{s \in \Gamma} p_s} \right) + \frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{(1 - \tilde{\pi})^2} \times \frac{\partial \tilde{\pi}}{\partial \alpha}
\end{aligned} \tag{45}$$

Using our fundamental equation, (16), and inserting its differential into (45) we have (after simplifying),

$$(r^{pr})'(\alpha) = -\frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{\sum_{s \in \Gamma} p_s} < 0 \tag{46}$$

Remark 9 The intuition behind (46) is follows: when allowing the expected-default level $\tilde{\pi}$ to vary, its variation $\partial \tilde{\pi} / \partial \alpha$ (which is negative by the first inequality of Proposition 2) is sufficiently negative such that it dominates $\frac{\partial r^{pr}}{\partial \alpha}$. Lenders who anticipate this, will require a lower premium due to the sovereign’s default rate being flexible. Notice, though, that when the default level is kept fixed, the impact of α on the default risk premium is positive, as shown below in (47). This is due to the fact that $1 + r$ increases with α , and its increase exceeds that of $1 + i$.

In what follows we shall analyze α derivatives when we keep either the expected-default level $\tilde{\pi}$ or α constant.

$$\frac{\partial r^{pr}}{\partial \alpha} = \left(\frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{1 - \tilde{\pi}} - \frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{\sum_{s \in \Gamma} p_s} \right) \equiv \frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{\sum_{s \in \Gamma} p_s} \left(\frac{\tilde{\pi} - (1 - \sum_{s \in \Gamma} p_s)}{1 - \tilde{\pi}} \right) > 0 \tag{47}$$

$$\frac{\partial r^{pr}}{\partial \tilde{\pi}} = \frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{(1 - \tilde{\pi})^2} > 0 \tag{48}$$

That is to say, the default risk premium is increasing in the size of the fiscal disturbance and in the probability of default. To enhance our intuition we can take a step back and begin by analyzing the earlier case in which the fiscal shock, α , was exogenous. For that, we rewrite (43) using (18) for the case of two states: a boom with probability p and a bust with probability $1 - p$.

$$r^{pr} = \frac{(\alpha - 1)(\gamma - (1 - p))}{1 - \gamma} \quad (49)$$

Denoting γ as the default level, condition (49) allows us to also incorporate the two familiar limiting cases of complete and no-default.

On one hand, making $\gamma \in [1 - p, 1]$ vary, we obtain something similar to the geometry of Section 5.4. That is to say, when $\gamma = 1$ we have the red vertical line $\alpha = 1$ denoting complete default; and when $\gamma = 1 - p$, the blue horizontal no-default line.

Therefore, see Figure 7, in the plane (α, r^{pr}) we plot all lines associated to each partial default level γ . We illustrate (47) and (48) for our special case of two states of nature: an increase in the shock, $\alpha \rightarrow \alpha'$ increases the premium from point $a \rightarrow b$. Similarly for an increase in the default probability, $r_{\gamma}^{pr} = p \frac{\alpha - 1}{(\gamma - 1)^2} > 0$, shown by the movement $a \rightarrow c$.

7.3 How Habits Modify the Default Risk Premium

We can now use the previous analysis to compute the impact that fiscal habits have on the default risk premium once we have not α but $\alpha(\rho) = \frac{1}{-E_0} \frac{\eta_1 - \rho \eta_0}{1 - \rho}$. In a partial default regime $\gamma \in (1 - p, 1)$, we have

$$r^{pr} = \frac{\gamma - (1 - p)}{1 - \gamma} m(\rho) \quad (49')$$

where $m(\rho) \equiv \alpha(\rho) - 1$. Note that this implies that the level curves, for each default level γ , are no longer linear (in fact, they are hyperbolas) except for the limit cases of $\gamma = 1, 1 - p$. This reflects the non-linearity of $m(\cdot)$, which is decreasing under bad habits $\eta_0 > \eta_1$.

From (49'), r^{pr} can be viewed as a function of $(\rho, \tilde{\pi})$. Hence, performing a similar analysis to that in (44), we see that along of the curve level $\gamma \in (1 - p, 1)$ the effect of habits is,

$$\frac{\partial r^{pr}}{\partial \rho} = \frac{\partial r^{pr}}{\partial \alpha} \alpha'(\rho) = \frac{\gamma - (1 - p)}{1 - \gamma} m'(\rho) < 0 \quad (50)$$

In other words, under bad habits as $\rho \rightarrow 1$, with the default level fixed, implies the movement $a \rightarrow b$.

Note let us permit

For the sake of completeness, we illustrate the previous facts in Figure 8 for the case of bad fiscal habits. When the probability of default increases for a given fiscal shock size, the premium increases: $c \rightarrow d$.

We finish this section we allow the default level γ to vary. In that case, the default risk premium will only be a function of the memory ρ . Being so, we can compute the impact of ρ on r^{pr} . We do it by applying the chain rule in (46) to get

$$(r^{pr})'(\rho) = (r^{pr})'(\alpha)\alpha'(\rho) > 0 \quad (51)$$

The positive sign in (51) states that under bad fiscal habits lenders demand a higher premium if the sovereign anchors her actions in the past (i.e. if sovereign increases her memory).

8 The Cost of Default, $\mathcal{C}$

Earlier we defined the case that sovereigns could form bad or good fiscal habits. Bad fiscal habits, to recall, equates to the sovereign's insistence on anchoring fiscal habits in the past. Accordingly, we might ask: under this regime, is it optimal for the government to increase its "memory"? The objective of this section is to answer this question.

For that we need to offer the government some mechanism that achieves its objective since our model does not consider optimizing governments per se. Our sovereign only enters the economy to balance its budget. However, since we are assuming defaulting governments, we can determine the cost incurred in defaulting.

Loosely speaking, the government finances its expenditure either by issuing debt or levying taxes. If the government defaults, its expenditures would be affected relative to that of a situation of no default. Since in our model both expenditures and taxes are exogenous and compacted in $E_0 < 0$, then the cost of default would fall entirely on the value of the debt.

Combining (1) and (22) we have that, on the one hand, the value of the debt $q\varphi$ satisfies

$$q\varphi = \frac{\varphi}{1+r} = -E_0 \quad (52)$$

On the other, if the government issued the same amount of bonds φ at price $1/(1+i)$, then the value of

the debt would be

$$\frac{\varphi}{1+i} = -E_0 + \mathcal{C} \quad (53)$$

since $1+i < 1+r$. This follows from the fact that

$$1+r = \frac{(1+i) \sum_{s \in \Gamma} p_s}{1-\tilde{\pi}} \quad (54)$$

which in turn follows after combining (16) and (18) under a partial-default regime. Equation (53) says that the government's leverage is financed by the issuance of bonds without default risk while in (52) the financing is carried out via the issuance of default-risk bonds. Adapting the analysis of Davydenko et al (2012) for firms, $\mathcal{C}$ is defined to be the cost of default incurred by the defaulting sovereign. Subtracting (52) from (53) and solving for $\mathcal{C}$ we have

$$\mathcal{C} = \varphi \left(\frac{1}{1+i} - \frac{1}{1+r} \right) \quad (55)$$

Manipulating in a convenient way inside of parenthesis of the previous equality and after using (1) we have

$$\mathcal{C} = \frac{-E_0}{1+i} r^{pr} \quad (56)$$

and using (43) we write the cost of default in terms of the expected default rate:

$$\mathcal{C} = -E_0 \left(\frac{\tilde{\pi} - (1 - \sum_{s \in \Gamma} p_s)}{1 - \tilde{\pi}} \right) \quad (57)$$

8.1 Optimal Fiscal Habits and the Cost of Default

From (57) we have that

$$\mathcal{C} = -E_0 \frac{\gamma - (1-p)}{1-\gamma} \quad (58)$$

Under a partial-default regime $\gamma \in (1-p, 1)$ from (15) follows that the cost of default in term of fiscal disturbances α is

$$\mathcal{C} = -E_0 \left(\frac{1+r}{\alpha-1} - 1 \right) \quad (59)$$

We can determine the fiscal effort or memory which minimizes the cost of default $\mathcal{C}$ once $\alpha(\rho) =$

$\frac{1}{-E_0} \frac{\eta_1 - \rho \eta_0}{1 - \rho}$. So, writing the cost of default $\mathcal{C}$ in terms of memory parameter ρ we have

$$\mathcal{C}(\rho) = -E_0 \left(\frac{1+r}{\alpha(\rho) - 1} - 1 \right). \quad (60)$$

Computing the derivative with respect to ρ :

$$\mathcal{C}'(\rho) = \frac{E_0(1+r)\alpha'(\rho)}{(\alpha(\rho) - 1)^2} > 0 \quad (61)$$

since $\alpha(\rho)$ is a decreasing function. Thus, the cost of default under bad fiscal habits is an increasing function so that it attains its minimum at $\rho_{min} = \frac{[\eta_1 + 2E_0]^+}{\eta_0 + 2E_0}$ as $E_0(p, 1 - p) = E_0$.

The intuition behind this result is that under bad habits, it is never optimal for the sovereign to increase its memory. Rather it should move away from the past decreasing the fiscal habit parameter until reaching ρ_{min} .

9 A Summary

Table 1 reprises some key qualitative predictions of the model. In the top half, the default environment is exogenous. For convenience, we simply sign these rather than provide the appropriate condition. Following Proposition 1, an increase in the disturbance (reflecting lower deficits for a fixed $\bar{A}$) reduces the probability of default and increases interest costs.

In the bottom half of the table the derivatives are instead a function of the habit regime (i.e., of η_0, η_1, ρ), with the result that $\bar{A}$ (or the fiscal effort) becomes endogenous. The qualitative behavior of the model is thus no longer unique, nor necessarily linear. We illustrated these outcomes under ‘bad’ habits. Accordingly, an increase in memory lowers fiscal effort and increases the probability of default (and lowers interest costs). The effect on $\bar{A}$, moreover, is negative and non linear (as indicated by “ Θ ”) reflecting the presence of ρ itself in the derivative: the higher the memory process, the more negative the effect on the boundary and the more positive effect on the probability of default.

Note also that – consistent with the “debt intolerance” view – the relationship between the deficit level and default is not clear cut. A large first-period fiscal burden complicates solvency since it requires sufficiently favorable second period outcomes. Thus, indeed, sovereign default can occur somewhat in proportion to the high levels of accumulated debt. On the other hand, if E_0 is by some measure small, then what will matter for the default event will be the size of the shock. In any case, in contrast to the

traditional literature on sovereign default, we postulate that default occurs by fiscal rather than purely economic questions (e.g., like, severe economic contractions): namely what form of habits the sovereign adopts.

Sovereign default occurs due to low future primary surpluses; and this is potentially due to bad fiscal behavior exercised by the government as well as the amplitude of fiscal shocks. We also cover this in the section dedicated to “endogenizing fiscal shocks”. Another point that our paper also would account for what the literature says is that despite the normal economic situation, the sovereign default occurs due to a change in the level of impatience of the government: it decides not to pay the outstanding debt in a given period in order to allocate that money to others current expenses. This situation is captured by the fiscal shocks assumed in Assumption 2.

10 Conclusions

Following the insight of Reinhart and Rogoff (2004), we studied the problem of a sovereign that defaults with some regularity, like Argentina. In practice of course there are many reasons why sovereigns default, and why some do so more often than others. Our contribution was specifically to consider sovereign default through the lens of uncertainty and fiscal habits. In so far as the national authorities lack financial discipline and suffer volatility, that may over time create fiscal habits with respect to its primary surpluses, since actual surpluses systematically differ from planned, expected surpluses. Moreover, we distinguished between good and bad habits. This allows us to have potentially different outcomes across similarly-indebted sovereigns.

We constructed a two-period general equilibrium model with this type of government and a representative lender. In equilibrium, lenders learn how to deal with borrowers with bad fiscal habits. The paper shows how habits influence the asset market clearing condition. Default occurs with a positive probability, which depends on the financial market conditions, on the interaction between the lender and the government, and on uncertainty.

Even taking into account a government that defaults systematically, the probability of default *is not necessarily equal to one*. The impact of habits on the expected default rate is the opposite of its effect on the interest cost of debt. It has a positive, zero or negative on the probability of default depending on the habit formation formed. Moreover, our framework can also be informative about which model elements

either expand or contract the available space for partial default to occur.⁴⁷

Possible policy implications of habits in sovereign debt management, include serial defaulters committing themselves to (or being held to) relatively lower fiscal ratios (a point made by Reinhart et al. 2003), or (as a commitment device) to issuing more longer-dated debt. There may also be a role for the promotion of non-partisan fiscal councils that inform the debt debate and independently scrutinize policy – prominent examples include the CBO in the US, the OBR (UK), the CPB (Netherlands), AIREF (Spain).

Future research can expand our model in several directions. Indeed, we hope policy habits makes for a promising research agenda, helping shed light on repeated sovereign default. It also arguably constitutes a more reasonable halfway house between weak “bad-luck” involuntary default assumptions on one hand and strong assumptions about voluntary default (i.e., governments make complicated welfare comparisons across default regimes) on the other. One step in that research agenda would be to integrate policy habit formation into quantitative models of sovereign default. Many such models have difficulty replicating key aspects of sovereign defaults e.g., the probability of default, the debt level at which default occurred, the output costs and so on (see the discussion in Stähler, 2013). These difficulties may be amenable to the implementation of habits.

A second extension could be the explicit modeling of an optimizing government playing as leader or follower with international lenders, yielding a framework that can be studied in an infinite horizon context as a differential game (Miller and Salmon, 1985) with potentially some new insights (such as the existence of cycles of default and re-payment, as well as agents’ time inconsistency). Moreover, serial defaulters often suffer currency crises, in which investors fear that servicing of foreign-currency denominated debt would be interrupted by exchange controls, yielding capital flight and the cutoff of hard-currency bank finance (Krugman, 1991). Such a crisis inevitably involves the domestic central bank, so the model could be extended by incorporating a (potentially independent) monetary authority trying to control inflation, unemployment and/or the exchange rate.⁴⁸

Next, most sovereign defaults end with settlements of some sort with creditors, so an additional avenue of inquiry is to study the essential role played by the bargaining power of the agents in these negotiations (Yue, 2010). Likewise, we might consider sovereigns having access to multiple lenders, thus

⁴⁷ See our analysis in Appendix G. When $S = 2$ or 3 the line of non-default is always 1. So when there is more uncertainty the slope of the line of non-default varies making the cone changes also. But despite that, regardless of whether it is acute or not, the partial-default cone contains all the default levels, only tighter, when the cone is acuter.

⁴⁸ On the connection between sovereign default and exchange rate devaluations see Na et al. (2018).

opening up possibilities for arbitrage and preferred-creditor considerations.

Finally, we assumed generic habits but, akin to the deep-habits literature (Ravn et al., 2006), we might also consider sovereigns having idiosyncratic habit-formation preferences over specific components of the primary surplus.

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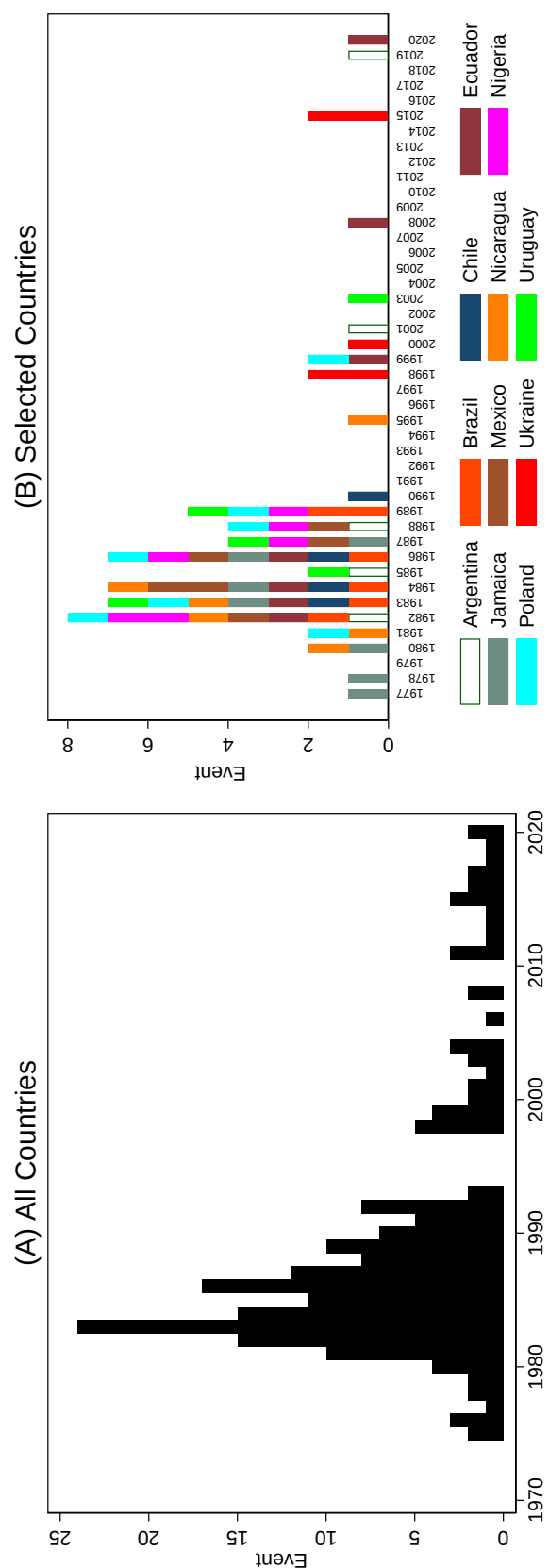
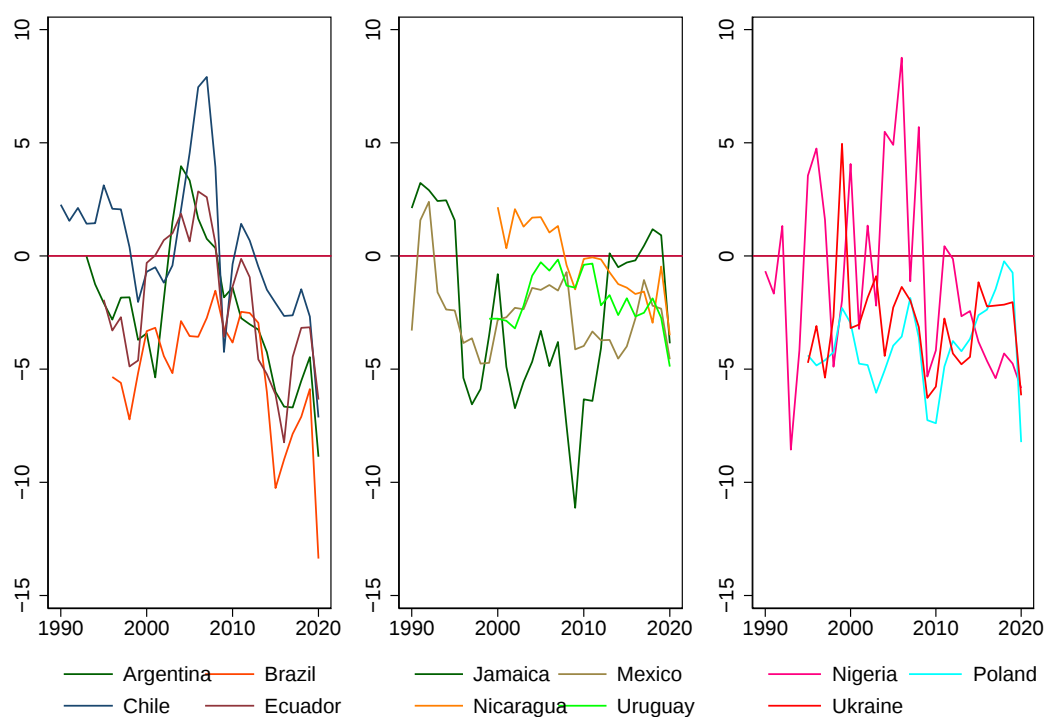


Figure 1: Start of Default Restructuring Process
(Default or Announcement)

Notes: The lhs graph registers any single default episode as a '1' among all the sampled countries from 1970 to 2020. The right side graph concentrates on countries for which there is at least five such episodes over the full sample.

Source: Updated database of Asonuma and Trebesch (2016), available at <https://sites.google.com/site/christophTrebesch/data>.

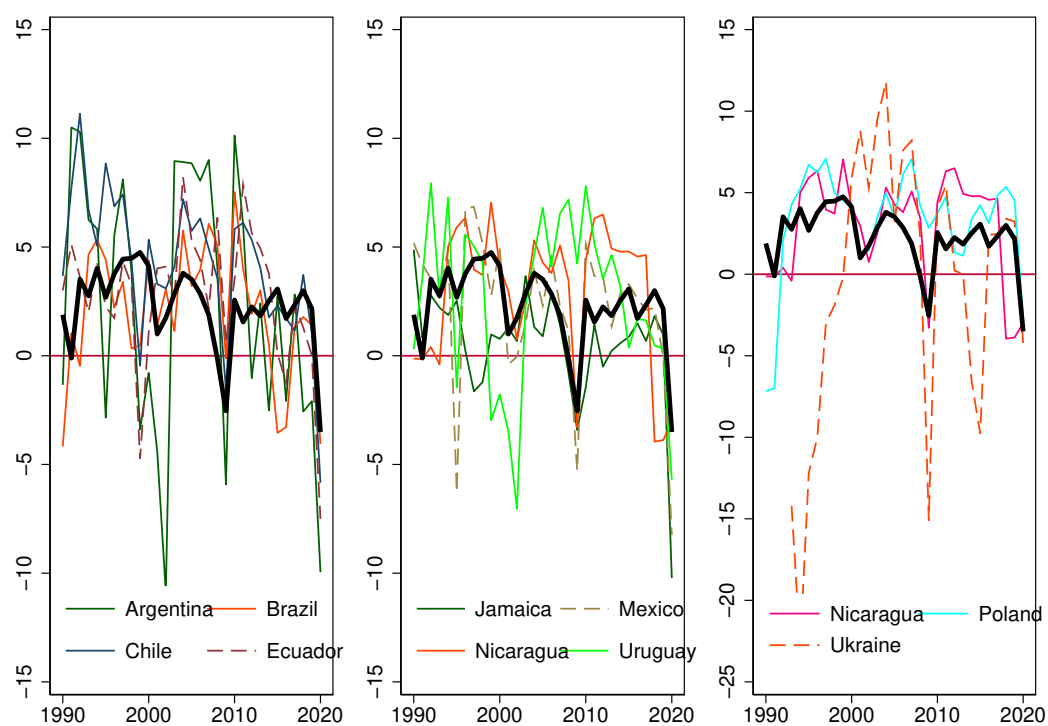
Figure 2: General Government Primary Net Lending/Borrowing



Notes: Annual Net lending (+)/ borrowing (-) is calculated as revenue minus total expenditure, as a percentage of GDP.

Source: IMF WEO April 2021.

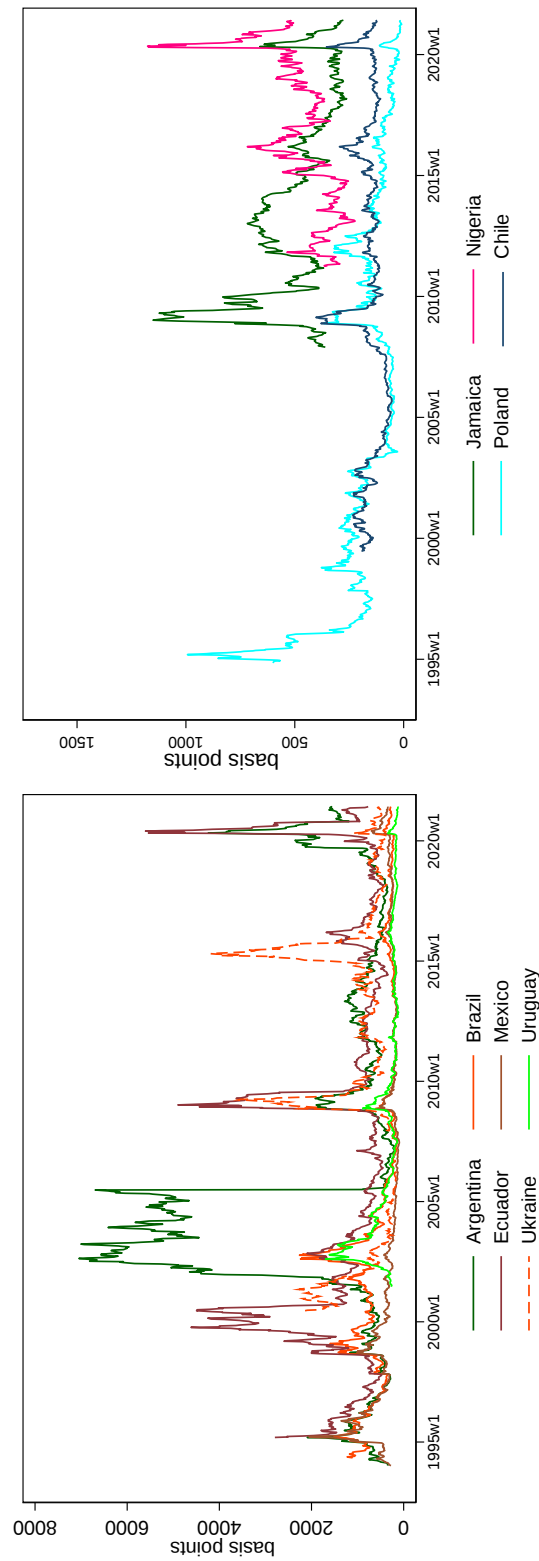
Figure 3: Annual Growth Rates of Selected Serial Defaulters, plus US



Notes: Annual Real Growth Rates % change from previous year. US series is overlaid as black thick line.

Source: IMF WEO April 2021.

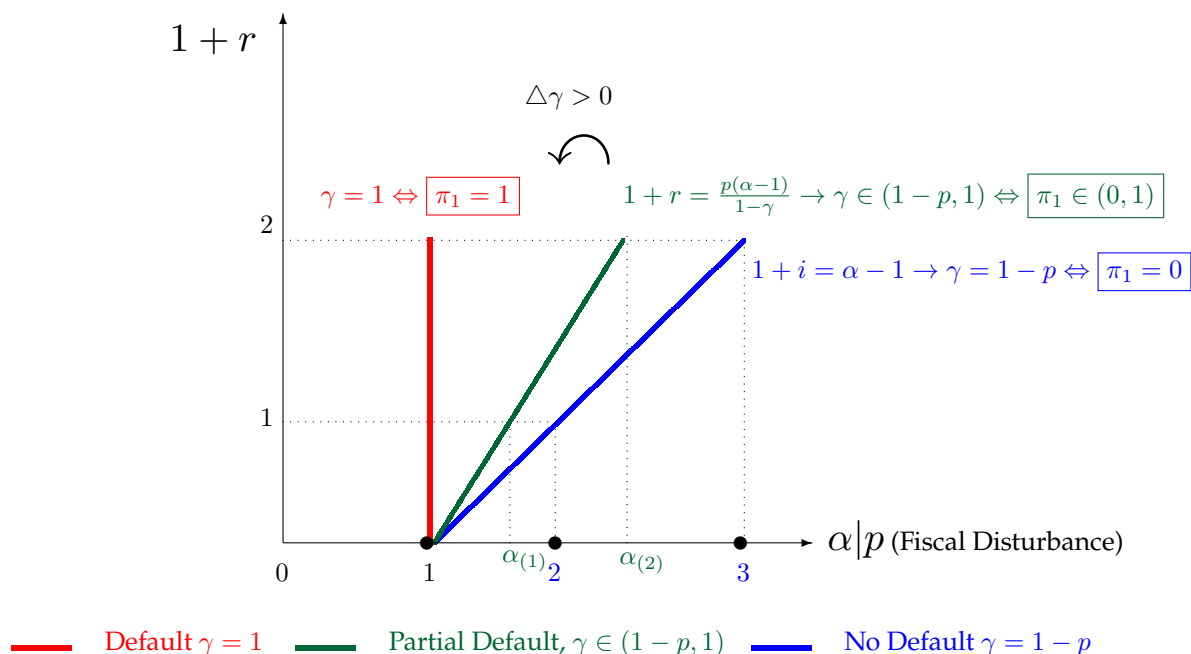
Figure 4: Sovereign Bond Spreads of Selected “Serial Defaulters”, basis points.



Notes: This figure shows the Emerging Market Bond Indices sovereign spread (measured in basis points). The “w” in the time axis stands for week since these daily data were converted to weekly averages for graphical convenience. The maximum sample runs from 1993w1 until 2021w23. Note the different vertical scales in each plot. Data for Nicaragua are not available in this dataset.

Source: JP Morgan/Haver Analytics.

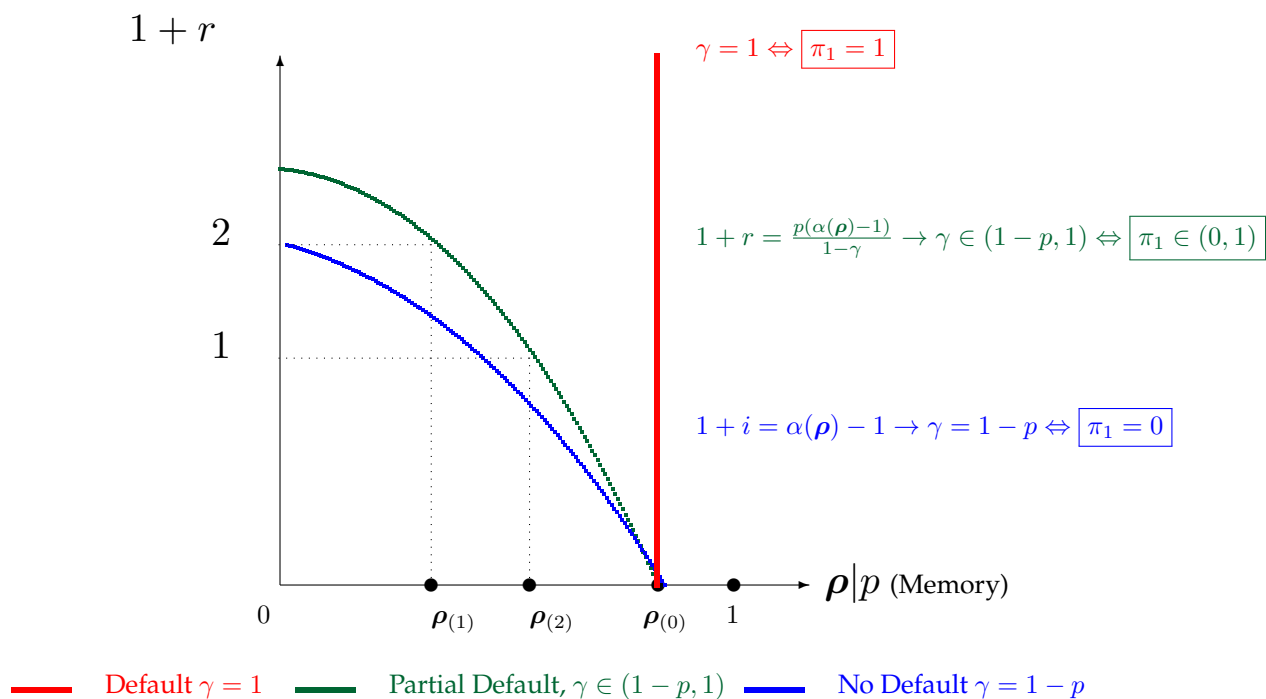
Figure 5: Default and No-Default Scenarios with Fixed Probabilities of a Boom p



Notes: For $S = 2$, and with $p_1 = p$ denoting the given probability of a boom, the probability of default can be viewed as a function of the shock and gross interest: $(\alpha, 1+r)$. We plot the gross nominal rate on a $1-2$ support, and all the level curves for each default level $\tilde{\pi} = \gamma \in [0, 1)$. The blue positively-sloped and red vertical line represent respectively no-default and complete default. Intermediate partial default cases are indicated by the anticlockwise arrow and the green line. The interval $[\alpha_{(1)}, \alpha_{(2)}]$ contains all values of α for which $1+r \in [1, 2]$ and for which there is a partial default, as shown in (21). The $\alpha = 2, 3$ are the α values for the no default case corresponding to the $1+r$ ranges.

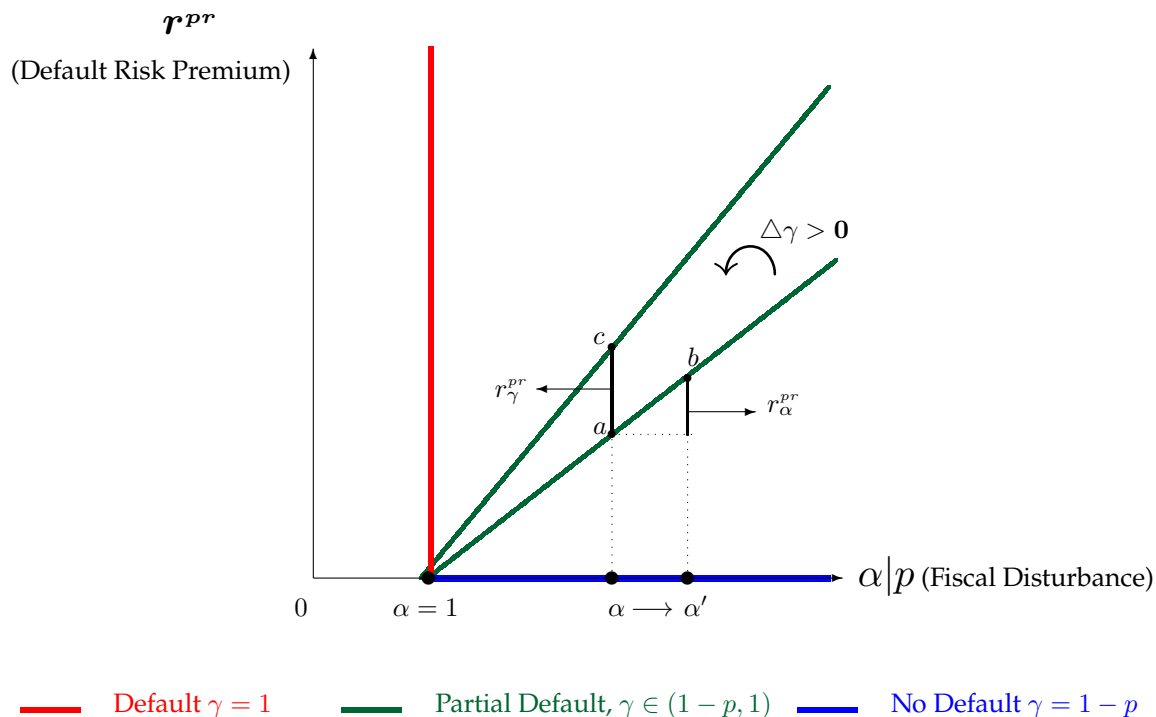
For a fixed level of default $\gamma \in [0, 1)$ and a fixed probability of a boom represented by p , the gross rate of interest is only determined by α which incorporates the government's leverage and fiscal shocks. Geometrically this is represented by all upward-sloping lines.

Figure 6: Default Regions Depending on Habit Parameter ρ with Fixed Probabilities of a Boom p



Notes: This figure restates Figure 5 for endogenous fiscal habits. The analysis is done in the plane $(\rho, 1+r)$ with a fixed p . Both the no- and partial default schedules are now curves instead of being straight lines due to non-linearity of the fiscal disturbance α . The partial default region is only parameterized by the probability of a boom p which is fixed. The interval $[\rho_{(1)}, \rho_{(2)}]$ is the feasible domain for each curve $1+r \in [1, 2]$. Lastly, all contour lines can be parameterized using two equivalent scales. Namely, via the expected default rate γ or via the default level in state 1: π_1 .

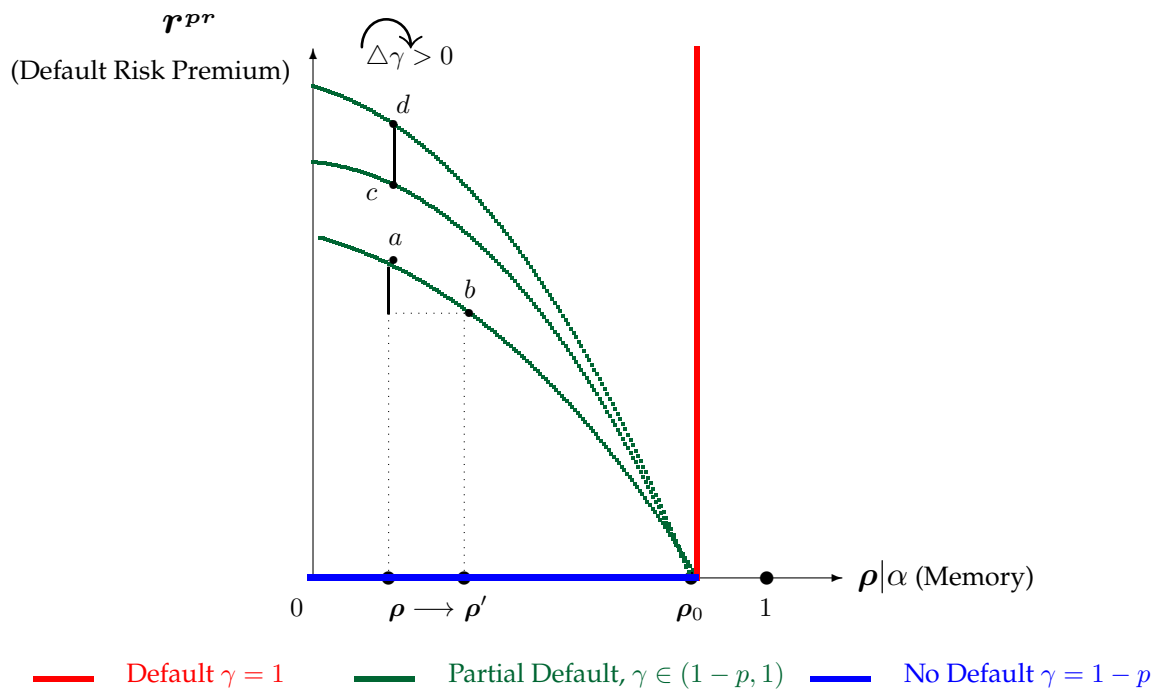
Figure 7: Sovereign Default Risk Premium and Fiscal Shocks, with fixed probabilities of a Boom p



Notes: For $S = 2$ and a given state-of-nature probability, we consider an increase in the fiscal shock from $\alpha \rightarrow \alpha'$. This increase may come from either an exogenous innovation or through the operation of a particular habit formation combination. The increase leads to an increase in the default risk premium by r_{α}^{pr} indicated by the $a \rightarrow c$ movement.

The figure also shows the impact on the default premium for a change in the default level $\Delta\gamma > 0$, as indicated by the curved arrow. This leads to a steepening in the slope of the partial-default schedule and an increase in the default risk, for a given fiscal-shock level α , where $r^{pr'} = \frac{\gamma + \Delta\gamma}{1 - (\gamma + \Delta\gamma)} p(\alpha - 1)$.

Figure 8: Default Risk Premium and Boom Probabilities
(‘Bad’ Fiscal Habits)



Notes: For $S = 2$ and bad endogenous fiscal habits i.e., $\eta_0 > \eta_1$ with $p_1 = p$ denoting the probability of a boom, this figure shows how the curve levels of the default risk premium $r^{pr}(\cdot)$ tend to the blue vertical line and green horizontal line representing complete default and no default respectively. ρ_0 is given by where $\rho_0 = \frac{\eta_1 + E_0}{\eta_0 + E_0} < 1$.

In addition, this figure also shows the impacts produced on the default risk premium due to the changes both of fiscal-habit parameter, ρ , and default-probability parameter, γ . These are represented by the black vertical segments.

Table 1: Qualitative Model Summary

	Disturbance	Deficit	Probability of	Cost of	Partial default
		Boundary	Default	Debt	Conditions
	$\alpha = -\frac{\bar{A}}{E_0}$	$\bar{A}$	$\tilde{\pi}$	$1 + r$	$1 - p < \tilde{\pi} < 1$
EXOGENOUS: NO FISCAL HABITS					
Disturbance, $\alpha \alpha > 1$	1		—	+	$1 < \alpha < 2$
Boom Probability, p_s			—	+	
ENDOGENOUS: FISCAL HABITS					
Habit Parameter: ρ	$sgn(\eta_1 - \eta_0)$	$\frac{\eta_1 - \eta_0}{(1 - \rho)^2}$	$-\frac{\sum_{s \in \Gamma} p_s \alpha'(\rho) \mathcal{I}_s}{1 + r}$	$\frac{\sum_{s \in \Gamma} p_s \alpha'(\rho) \mathcal{I}_s}{1 - \tilde{\pi}}$	
For $\eta_0 > \eta_1$	—	$\ominus$	+	—	$\rho_{min} < \rho < \rho_{max}$
Boom Probability, p_s			$-\frac{\alpha(\rho) \mathcal{I}_s - 1}{1 + r}$	$\frac{\alpha(\rho) \mathcal{I}_s - 1}{1 - \tilde{\pi}}$	
For $\eta_0 > \eta_1$			—	+	

Notes: This table shows the qualitative signs of selected derivative relationships embedded in the model framework. In the first row we consider the effect of an increase in the fiscal disturbance conditional on $\alpha > 1$. In the second row we consider the comparative statics of an increase in the probability of a boom. In the second half of the table we delineate the derivatives of a change in the habit parameter and then by the probability of a boom. Below these two rows respectively we show the sign of the derivative under bad fiscal habits. An empty cell indicates no relationship, or no direct relationship. A circle over the sign indicates a non-linear relationship.

A Selected Proofs

Proof of Proposition 1

Assumptions 1 and 2 imply that

$$1 - \pi_s = \frac{E_0 + \bar{A}\mathcal{I}_s}{-(1+r)E_0}, s \in \Gamma \quad (\text{A.1})$$

Using the definition of α and after substituting it into Item 1 of Lemma 1, (16) follows. Since $E_s = E_0 + \bar{A}\mathcal{I}_s$ is always positive at states belonging to Γ , it follows that the right side of (16) is strictly positive, implying $\tilde{\pi} < 1$. Hence, using Item 3 of Lemma 1 the equilibrium is partial: viz, $\exists s \in \Gamma : \pi_s < 1$.

Finally, arranging (14) in a convenient way, we have that for any $r \geq 0$ the following holds true $\frac{\sum_{s \in \Gamma} p_s \alpha \mathcal{I}_s}{\sum_{s \in \Gamma} p_s} < 2 < 1 + (1+r)$ which implies that $\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1) < (1+r) \sum_{s \in \Gamma} p_s$. Dividing by $(1+r)$ on both sides of the previous inequality and using (16) we have $1 - \tilde{\pi} < \sum_{s \in \Gamma} p_s$. This implies that $\tilde{\pi} \neq 1 - \sum_{s \in \Gamma} p_s = \sum_{s \in \Gamma'} p_s$. Hence, Item 4 of Lemma 1 implies that $\exists s \in \Gamma : \pi_s > 0$. Thus there exists default in equilibrium so that it is non trivial according to Definition 2.

Proof of Corollary 1

Follows from (17) after substituting $\frac{\sum_{s \in \Gamma} p_s}{\sum_{s \in \Gamma} p_s \mathcal{I}_s}$ for α .

Proof of Proposition 2

The former inequality can be obtained after differentiating (16) with respect to α . In effect,

$$\frac{\partial \tilde{\pi}}{\partial \alpha} = - \frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{1+r} \quad (\text{A.2})$$

which is clearly negative. Since we are in a partial-default regime we can solve $1+r$ from (16). Differentiating with respect to α , the latter inequality in (19) follows.

Proof of Corollary 2

By arranging and simplifying (18) we have

$$1 + i = \frac{\sum_{s \in \Gamma} p_s \mathcal{I}_s}{\sum_{s \in \Gamma} p_s} \alpha - 1 \quad (\text{A.3})$$

Differentiating (A.3) with respect to $p_{s'}$ with $s' \in \Gamma$, we have

$$\frac{\partial(1+i)}{\partial p_{s'}} = \frac{\sum_{s \in \Gamma} p_s (\mathcal{I}_{s'} - \mathcal{I}_s)}{(\sum_{s \in \Gamma} p_s)^2} \quad (\text{A.4})$$

If $s' = 1$ Item 1 follows since $\mathcal{I}_s < 1$. If $s = \lceil S/2 \rceil$ Item 2 follows as the function $\mathcal{I}_s$ is strictly decreasing in s . Finally, if $1 < s' < \lceil S/2 \rceil$, then Item 3 follows as the numerator in (A.4) contains negative and positive real numbers each of them weighted by $p_s \in (0, 1)$ which varies in intensity.

Proof of Lemma 3

Lemma 3 follows directly after substituting (25) into (14) and solving for ρ .

Proof of Corollary 3

For the first relation, differentiate (35) with respect to ρ , keeping fixed $1 + r$. For the second, do the same as before but now, keep fixed $\tilde{\pi}$.

B Existence of Equilibrium

B.1 Truncated Economies and Associated Generalized Games

If there was trading and complete default at every state of nature, the price of the public debt would obviously be zero. Because of this fact, the existence argument is somewhat intricate. In the spirit of Geanakoplos and Zame (2014), we construct a sequence of economies $\mathcal{E}^n$ for which the price of public debt is $q = \frac{1}{1+r} > \frac{1}{n}$; so that $r \in [0, n-1] \Leftrightarrow q \in [\frac{1}{n}, 1]$. Then, we construct an equilibrium for $\mathcal{E}$ by taking limits as $n \rightarrow \infty$.

Consider a sequence of truncated economies $\{\mathcal{E}^n\}_{n \geq 1}$ in which the budget set of the lender is

$$B_n(q, \pi) = \{(c, \theta) \in [0, n]^{1+S} \times [0, n] : \text{equations (3) and (4) are satisfied}\} \quad (\text{B.1})$$

where $\pi \in [0, 1]^S$. In addition, the government chooses $(\varphi, D) \in [0, -nE_0] \times [0, n]^S$ such that equations (1), (2) and (3) are satisfied.

Define z to be

$$[(c^h, \theta^h); (\varphi, D); q, \pi] \in (\mathbb{R}_+^{1+S} \times \mathbb{R}_+) \times (\mathbb{R}_+ \times \mathbb{R}_+^S) \times \mathbb{R}_+ \times \mathbb{R}_+ \quad (\text{B.2})$$

and $z_{-\kappa}$ the vector z in the κ -coordinate has been dropped. The sub-index, κ , is any coordinate of the vector z just defined.

For each $n \geq 1$ we define the following generalized game played by 4 players. We denote this game by $\mathcal{G}^n$ which is described as follows:

1. The first player is the lender whose payoff is their utility function $u^h : \mathbb{R}_+^{1+S} \rightarrow \mathbb{R}$ and its strategy set is $B_n(q, \pi)$.
2. The second player, representing the government, chooses $(\varphi, D) \in [0, -nE_0] \times [0, n]^S$ in order to maximize its payoff, denoted by $F(\cdot)$

$$F((\varphi, D); z_{-(\varphi, D)}) = -(q\varphi + E_0)^2 - \sum_{s=1}^S (D_s - E_s^+)^2$$

3. The third player is an auctioneer who chooses $q \in [\frac{1}{n}, 1]$ in order to maximize its payoff, denoted by $\ell(\cdot)$

$$\ell(q, z_{-q}) = q(\theta - \varphi) + (c_0 - w_0 - E_0)$$

4. The fourth player is a fictitious agent (essentially the market mechanism) who chooses $\pi \in [0, 1]^S$

in order to maximize its payoff, denoted by $G(\cdot)$

$$G(\pi, z_{-\pi}) = - \sum_{s=1}^S (D_s - (1 - \pi_s)\theta)^2$$

We have the following definition:

Definition 4 *The array $[(q, \pi); (c, \theta); (\varphi, D)]$ constitutes an equilibrium in pure strategies for $\mathcal{G}^n$ if each player maximizes their strategy set given the actions of their rivals.*

Notice that each element of the earlier array depends on n , but we drop this sub-index in order not to accrue the notation. The following lemma guarantees the existence of an equilibrium for $\mathcal{G}^n$. Its proof follows from the equilibrium existence theorem in a generalized game of Debreu (1952).

Lemma 4 *Under the hypotheses of Theorem 1, the generalized game $\mathcal{G}^n$ has an equilibrium in pure strategies.*

The following lemma guarantees an equilibrium for each truncated economy $\mathcal{E}^n$.

Lemma 5 *For each n , a pure strategy equilibrium for $\mathcal{G}^n$ is also an equilibrium for $\mathcal{E}^n$.*

Proof of Lemma 5 To avoid cluttering notation, all indexes n of equilibrium variables of $\mathcal{G}^n$ will momentarily be suppressed. Let $[(q, \pi); (c, \theta); (\varphi, D)] \in [\frac{1}{n}, 1] \times [0, 1] \times B_n(q, \pi) \times [0, -nE_0] \times [0, n]^S$ be a pure strategy equilibrium for $\mathcal{G}^n$. From the definition of equilibrium point, the vector (c, θ) is budget feasible and maximizes $u(\cdot)$ on $B_n(q, \pi)$. Thus, item (i) of Definition 1 holds.

From the budget feasibility of the vector (c, θ) , it follows that equations (3) and (4), relabeled here as (B.3) and (B.4), are satisfied. More precisely,

$$c_0 + q\theta \leq w_0 \tag{B.3}$$

$$c_s \leq w_s + (1 - \pi_s)\theta \tag{B.4}$$

Optimal conditions of the government imply that

$$-E_0 = q\varphi \tag{B.5}$$

$$D_s = E_s^+ \tag{B.6}$$

so that the government balances its budget constraints.

Optimal conditions of fictitious agent who chooses π imply that

$$D_s = (1 - \pi_s)\theta \tag{B.7}$$

and therefore *item (iv)* of Definition 1 is satisfied. Moreover, the right side in (B.7) is less than θ and therefore $D_s \leq \theta$. This fact, together with (B.3) and (B.4) imply *item (ii)* of Definition 1.

It remains to prove *item (iii)* to end the proof of Lemma 4. In fact, the optimality conditions of the auctioneer's problem imply that

$$\theta - \varphi \leq 0 \quad \text{and} \quad (B.8)$$

$$c_0 - w_0 - E_0 \leq 0 \quad (B.9)$$

Adding (B.3) and (B.5) and knowing that u is strictly increasing, we obtain

$$q(\theta - \varphi) + (c_0 - w_0 - E_0) = 0 \quad (B.10)$$

Since $q \geq \frac{1}{n}$ is strictly positive, the terms of (B.10) are non-positive due to (B.8) and (B.9). Thus, each term in (B.10) is zero and therefore

$$q(\theta - \varphi) = 0 \quad (B.11)$$

Finally, *iii* of Definition 1 follows as $q > \frac{1}{n}$. That is,

$$\theta = \varphi \quad (B.12)$$

Thus, Lemme 5 follows.

B.2 Asymptotics of Truncated Equilibria and Proof of Theorem 1

To prove Theorem 1 we analyze the asymptotic properties of the sequence of equilibria

$$\{e^n = [(q^n, \pi^n); (c^n, \theta^n); (\varphi^n, D^n)]\}_{n \geq 1}$$

which exist from 5. Actually, we will demonstrate that the sequence of equilibria above is uniformly bounded, and therefore it will have a subsequence that converges, say $\bar{e}$. Theorem 1 will then be shown if we prove that $\bar{e}$ corresponds to an equilibrium of our original economy $\mathcal{E}$.

First, $q_n \in [0, 1]$ and $\pi_n \in [0, 1]^S$ for every n so that the sequence $\{(q_n, \pi_n)\}_{n \geq 1}$ is bounded uniformly.

Second, the fact that (c_n, θ_n) belongs to the budget set $B(q_n, \pi_n)$ implies that

$$\begin{aligned} c_{on} + q_n \theta_n &\leq w_0 \\ c_{sn} &\leq w_s + (1 - \pi_{sn}) \theta_n \end{aligned} \tag{B.13}$$

The first inequality of (B.13) implies that for each n , $c_{on} \in [0, w_0]$.

Third, due to the government balances its first-period budget constraint, we have

$$q_n \varphi_n = -E_0 \tag{B.14}$$

Equality (B.14) and (B.3) together with the fact that q_n belongs to $[\frac{1}{n}, 1]$ imply the following

$$\frac{\varphi_n}{n} \leq -E_0 \leq \varphi_n$$

and

$$\theta_n \leq w_0 n$$

We claim that $\varphi_n \leq -E_0$ for if we had the contrary, say $\varphi_n = -E_0 + \Xi$ for some $\Xi > 0$, we would contradict (B.8) for a n large enough. In fact, if we choose $\theta_n = nw_0$ and $\varphi_n = -E_0 + \Xi$, we would have,

$$\theta_n - \varphi_n = nw_0 + E_0 - \Xi$$

which for a n sufficiently large $\theta_n - \varphi_n > 0$, thus contradicting (B.8). Therefore, the sequences $\{\theta_n = \varphi_n\}_{n \geq 1}$ are uniformly bounded. That is, $\theta_n = \varphi_n \in [0, -E_0]$. Fourth, using this fact and the second inequality in (B.13), we have $c_{sn} \in [0, w - E_0]$ where $w = \max_s w_s$. Finally, from (B.2) it follows that the sequence $\{D_{sn}\}_{n \geq 1}$ is also uniformly bounded as $D_{sn} = (1 - \pi_{sn})\theta_n \leq \theta_n$. That is, $D_{sn} \in [0, -E_0]$.

From the earlier analysis, we have that the sequence $\{e^n\}_{n \geq 1}$ is uniformly bounded, so that it converges along a subsequence, say to $\bar{e} = [(\bar{q}, \bar{\pi}); (\bar{c}, \bar{\theta}); (\bar{\varphi}, \bar{D})]$. Next, we shall prove that $\bar{e}$ is an equilibrium for $\mathcal{E}$.

We state that the array $\bar{e}$ satisfies *conditions ii), iii) and iv)* of Definition 1. This follows after taking limits in equations (B.5), (B.6), (B.7) and (B.12). It remains to prove condition *i)* of Definition 1. For that, we require the following claim.

Claim The budget set correspondence $B : [0, 1] \times [0, 1]^S \rightrightarrows [0, w_0] \times [0, w - E_0]^S \times [0, -E_0]$ defined by $B(q, \pi)$ is lower hemicontinuous (lhc) at the point $(\bar{q}, \bar{\pi})$.

Proof of Claim

To start with, we state that the correspondence defined by

$$Int B(q, \pi) = \{(c, \theta) \in [0, w_0] \times [0, w - E_0]^S \times [0, -E_0] : \text{equations (4) and (5) hold with strictly inequality}\}$$

is lhc at the point $(\bar{q}, \bar{\pi})$. In fact, $Int B(p, \pi)$ is a non empty set, since $c = 0, \theta = 0$ satisfy equations (4) and (5) with strict inequality as $w_0 > 0, w_s > 0, \forall s$.

Let $\{(q_n, \pi_n)\}$ be a sequence converging to $(\bar{q}, \bar{\pi})$ and be $(c', \theta') \in Int B(p, \pi)$. Thus, for every sequence $\{(c'_n, \theta'_n)\}_{n \geq 1}$ converging to (c', θ') belonging to $Int B(\bar{q}, \bar{\pi})$ we have that for n large enough (c'_n, θ'_n) belonging to $Int B(q_n, \pi_n)$ which implies that $Int B(\cdot)$ is lhc.

Next, from Hildenbrand (1974, p.26, Fact 4), it follows that the correspondence $B(\cdot)$ defined in the earlier claim, which is the closure of $Int B(\cdot)$, is also lhc. Thus, the Claim follows.

To prove *condition i*), let us suppose the contrary. That is, $(\bar{c}, \bar{\theta})$ does not maximize u subject to $B(\bar{q}, \bar{\pi})$ being $(\bar{q}, \bar{\pi})$ a the cluster point of $\{(q_n, \pi_n)\}_{n \geq 1}$. Hence, there exist $(c, \theta) \in B(\bar{q}, \bar{\pi})$ satisfying $u(c) > u(\bar{c})$. The early claim implies that there exists a sequence $\{(c'_n, \theta'_n)\}_{n \geq 1} \subset B(q_n, \pi_n)$ such that $(c'_n, \theta'_n) \rightarrow (c, \theta)$. Notice that the arguments of $B(\cdot)$ are terms of the sequence that form part of the sequence of equilibria $\{e^n\}$ of the truncated economy. That is, $(q_n, \pi_n) \rightarrow (\bar{q}, \bar{\pi})$.

Condition ii) of Assumption 1 implies that u is continuous so that for n large enough, we obtain $u(c'_n) > u(c_n)$ contradicting the optimality of c_n in the truncated economy $\mathcal{E}^n$. This ends the proof of existence. To finish the proof of Theorem 1, we are going to show that there is trading.

From (B.14) (in the limit) and i) of Assumption 1 imply that $\bar{q}\bar{\varphi} = -E_0 > 0$. This fact implies that both $\bar{q} > 0$ and $\bar{\varphi} > 0$. and thus that there is trading in equilibrium. That is, $\theta = \varphi > 0$.

C Equilibrium with Log Utility

The purpose of this example is to show that *Condition iv* of Assumption 1 is not sufficient for the default to exist in equilibrium. For that, consider an economy with $S > 1$ and $J = 1$, and two economic agents: a government and a representative lender. The government is characterized by its primary surplus $E = (E_0, E_1, \dots, E_S)$ where $E_0 < 0$ and $E_s > 0$ for some s . Therefore, the conditions in *i*) of Assumption 1 are satisfied. Moreover, we assume that the lender has logarithmic utilities. That is, the representative lender is characterized by

$$U(c_0, c_1, \dots, c_S) = \log c_0 + \sum_{s=1}^S \beta p_s \log(c_s)$$

and their initial endowments are $w = (w_0, w_1, w_2) >> 0$ where $w_0 > -E_0$. In addition, the probability of state s occurs, p_s , is assumed to be $0 < p_s < 1$. Although logarithmic utilities are not defined the boundary of the non-negative orthant as required in Assumption 1, we modify them, following Dubey et al. (2004), to meet the requirements of Theorem 1. So, by $\log(x)$ we shall mean

$$\log(x) = \begin{cases} \ln x, & \text{if } x > \delta \\ \frac{1}{\delta}x + \ln \delta - 1, & \text{if } 0 \leq x \leq \delta \end{cases}$$

for some very small $\delta > 0$. Here, $\ln(\cdot)$ denotes the natural logarithmic utilities. Thus, all assumptions of Theorem 1 are satisfied so that an equilibrium exists.

In what follows, suppose that there exists only one state such that $E_s > 0$. Without loss of generality, we assume only $E_1 > 0$ with $p_1 = p \in (0, 1)$. Lastly, let us define the payment rate $\tau_s = 1 - \pi_s$.

C.1 Equilibrium without Default

Suppose $0 \leq c_s \leq \delta, s = 1, \dots, S$. This implies that both $-E_0$ and E_1 are below δ . In this case the first order condition for the lender's problem are:

$$\sum_{s=1}^S \beta p \tau_s \leq q \quad \text{and} \quad \theta \left(\sum_{s=1}^S \beta p \tau_s - q \right) = 0. \quad (\text{C.1})$$

For the sake of completeness, we write the conditions (1), (2) and (3) that characterize (φ, D) .

$$0 = E_0 + q\varphi, \quad (\text{C.2})$$

$$D_s = E_s^+ \quad (\text{C.3})$$

$$0 \leq D_s \leq \varphi \quad (\text{C.4})$$

The equilibrium for this logarithmic economy is characterized by (C.1)-(C.4) and

$$\theta = \varphi \quad \text{and} \quad D_s = \tau_s \theta \quad (\text{C.5})$$

First, since $-E_0 < 0$ we have that the first equality in (C.2) and the first one in (C.5) imply $\varphi = \theta > 0, q > 0$. Second, since $E_s = 0$ for all $s \neq 1$ we have $\tau_s = 0$ as $\theta > 0$. On the other hand, $\tau_1 > 0$ as $E_1 > 0$. Therefore, (C.1) is reduced to

$$\beta p \tau_1 = q \quad (\text{C.6})$$

From (C.3) and the second equality in (C.5) we have

$$\tau_1 = \frac{qE_1}{-E_0} \quad (\text{C.7})$$

Combining (C.6) and (C.7) we have $\beta = -E_0/pE_1$. Since $\beta \in (0, 1)$, we have $-E_0 < pE_1 < E_1$. Putting $-E_0 = \epsilon E_1$ with $\epsilon \in (0, 1)$, depending $\delta > 0$, we have that for any $q = (1 + r)^{-1}$, we always find $\bar{\epsilon}$ such that $-E_0 = \bar{\epsilon} E_1 = qE_1$. Notice that we are assuming that $r \in [0, 1]$. However, even if r were greater than 1 and even if tending to infinity, we can always find an $\bar{\epsilon}$ such that $-E_0 = \bar{\epsilon} E_1 = qE_1$. This implies that $\tau_1 = 1$ provided that the leverage $-E_0$ is small in the sense that $-E_0 < E_1 \leq \delta > 0$.

C.2 Default in equilibrium

In this part, we are going to characterize the equilibrium, and show that to assume $w_0 + E_0 > 0$ is not suffice to guarantee default in equilibrium.

Suppose $c_s \geq \delta, s = 1, 1, \dots, S$. In this case the first order condition for the lender's problem are:

$$\frac{q}{w_0 - q\theta} = \frac{\beta p \tau_1}{w_1 + \tau_1 \theta}, \quad (\text{C.8})$$

since we are assuming that only $E_1 > 0$, which implies that both $\tau_1 > 0$ and $\theta > 0$ as claimed above.

In this case the equilibrium is characterized by (C.2)-(C.5) and (C.10).

Combining all these equations, we have that (C.10) can be written as,

$$\frac{q}{w_0 + E_0} = \frac{\beta p \tau_1}{w_1 + E_1} \Leftrightarrow q = \frac{\beta p \tau_1 (w_0 + E_0)}{w_1 + E_1} \quad (\text{C.9})$$

Solving for τ_1/q , we obtain

$$\frac{\tau_1}{q} = \frac{w_1 + E_1}{\beta p (w_0 + E_0)} \quad (\text{C.10})$$

Combining (C.2)-(C.5) we have

$$\frac{\tau_1}{q} = \frac{E_1}{-E_0} \quad (\text{C.11})$$

Equations (C.10) and (C.11) are two different ways of expressing the effective rate of return $\tau_1/q = (1+r)\tau_1$. Condition iv of Assumption 1 implies that $\tau_1 > 0 \Leftrightarrow \pi_1 < 1$. That is, if the default exists, it is not complete. However, $w_0 + E_0 > 0$ itself does not guarantee default in equilibrium. Next, we are going to give a sufficient condition, in terms of w_0 and the leverage E_0 , for this to occur.

Assumption 5 (Arbitrarily Large w_0) Define $P(w_0) \equiv w_0 + E_0 > 0$, and assume that $P(w_0)$ holds for an arbitrary large w_0 .

Mathematically, the above means:

$$\forall N, \exists w_0 \text{ such that } w_0 > N \text{ and } w_0 + E_0 > 0 \quad (\text{C.12})$$

For the sake of applicability, this condition is not satisfied if

$$\exists N, \forall w_0 \text{ such that } w_0 > N \Rightarrow w_0 + E_0 \leq 0 \quad (\text{C.13})$$

Combining (C.10) and (C.11) in a convenient manner, we obtain

$$\tau_1 = q \frac{w_1}{p\beta w_0 + E_0(1 + p\beta)} \quad (\text{C.14})$$

By definition $q = (1+r)^{-1} < 1$ as $r \in [0, 1]$. To prove that $\tau_1 < 1 \Leftrightarrow \pi_1 > 0$, and therefore default in equilibrium, it is sufficient that the fraction on the right side of (C.14) is strictly below 1.

Let us parameterize the lender by $\beta \in (0, 1]$ and $(w_0, (w_1, \dots, w_S)) \gg 0$ with w_0 satisfying Assumption 5. Since Assumption 5 holds, then putting $N = (-E_0)(1 + \frac{1}{\beta p}) + \frac{w_1}{\beta p}$ we conclude that for any lender with a discounted factor $\beta \in (0, 1]$ and every government with a leverage $-E_0 > 0$, there exists

w_0 , depending on β , w_1 , $-E_0$ and p ; such that

$$w_0 > -E_0 \left(1 + \frac{1}{\beta p_s} \right) + \frac{w_1}{\beta p} \quad \text{and} \quad w_0 + E_0 > 0 \text{ holds} \quad (\text{C.15})$$

The first inequality in (C.15) implies that the right side in (C.14) is strictly below 1. Therefore, default exists in equilibrium.

D Topological Robustness

Formally, the robustness of partial default follows from equation (16). In fact, if we solve $\tilde{\pi}$ from (16), we can consider $\tilde{\pi}$ as a function of $x = (p_1, \dots, p_S, 1 + r, \alpha)$. That is,

$$\tilde{\pi} = f(x) = 1 - \frac{\sum_{s \in \Gamma} p_s (\alpha \mathcal{I}_s - 1)}{1 + r} \quad (\text{D.1})$$

Clearly the expected default rate as a function of x is a differentiable and therefore continuous function. In particular, for the case $S = 2$ and α constant, the level curves of $f(\alpha, 1 + r)$ defined by (D.1) are given by (22). That is, $f(x) = 1 - \frac{p(\alpha-1)}{1+r}$.

Suppose that we are in a partial default $\gamma_0 \in (1 - p, 1)$ in equilibrium so that $f(\alpha, 1 + r) = \gamma_0$. Suppose that at that level γ_0 the economy is characterized by an interest rate r_0 and that the fiscal shock for α_0 : $f(\alpha_0, 1 + r_0) = \gamma_0$. Then, the continuity of f implies that, for any open interval $I(\gamma_0) \subset (0, 1)$ containing γ_0 , there exists an open ball $\mathbf{B}_\delta(\alpha_0, 1 + r_0)$ centered at $(\alpha_0, 1 + r_0)$ and radius $\delta > 0$ such that $f(\mathbf{B}_\delta(\alpha_0, 1 + r_0)) \subset I(\gamma_0) \subset (1 - p, 1)$. This means that for all $(\alpha, 1 + r)$ belonging to $\mathbf{B}_\delta(\alpha_0, 1 + r_0)$, we have that $f(\alpha, 1 + r) \in I(\gamma_0)$ and therefore the level $\gamma = f(\alpha, 1 + r) \in (1 - p, 1)$, implying a partial default. In others words, if the economy is in an equilibrium with a partial default, then, for any small perturbation of parameters $(\alpha_0, 1 + r_0)$, the economy will still be in an equilibrium with a partial default. This shows the topological robustness of the partial-default case.

D.1 The No-Default Limit Case

We begin by analyzing the case of no default $\gamma = 1 - p$ for the same particular case above. Suppose that at that regime the state of the economy is $(\alpha, 1 + r)$ satisfying $f(\alpha, 1 + r) = 1 - p$. In this case, $1 + r = \alpha - 1$. Take the sequence $(\alpha - \frac{1}{n}, 1 + r) \rightarrow (\alpha, 1 + r)$ as $n \rightarrow \infty$. Evaluating f at points $(\alpha - \frac{1}{n}, 1 + r)$ we have that the value of function at those points

$$f\left(\alpha - \frac{1}{n}, 1 + r\right) = 1 - \frac{p(\alpha - \frac{1}{n} - 1)}{1 + r} = 1 - \frac{p(\alpha - 1)}{1 + r} + \frac{p}{n(1 + r)} = 1 - p + \frac{p}{n(1 + r)} \quad (\text{D.2})$$

is strictly below 1. Thus, (D.2) represents a partial default. Now, taking the limit as $n \rightarrow \infty$ and thanks to continuity of f , (D.2) becomes $f(\alpha, 1 + r) = 1 - p$, which is the no-default regime. Hence, we have passed from a state of no default to a state of partial default by making small perturbations of the parameters (in this case of α). This shows that the case of no default is not robust in the topological sense.

D.2 The Complete-Default Limit Case

For the case of complete default $\gamma = 1$ we cannot use the function f above since it is not well defined at that level of default. As mentioned above, the vertical line $\alpha = 1$ is associated with $\gamma = 1$. Accordingly, the strategy will be to prove that this line vertical line is the limit of the lines associated to level curves of f in an appropriate topology where the lines are considered as points of some topological space: namely, some quotient space. Next, we are going to elaborate this argument to reach our objective.

Consider again, function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by (D.2). For any $\gamma \in \mathbb{R}$ define the set $L_\gamma := f^{-1}(\gamma)$ to be

$$L_\gamma := f^{-1}(\gamma) = \left\{ (\alpha, 1+r) : 1 - \frac{p(\alpha-1)}{1+r} = \gamma \right\} \subset \mathbb{R}^2$$

Geometrically, for $\gamma \neq 1$, L_γ is a straight line laying into $\mathbb{R}^2$ whose cartesian equation is given by (22) after solving for γ . For $\gamma = 1$, L_1 is the vertical line

$$\alpha = 1 \tag{D.3}$$

Notice that for any $(\alpha, 1+r) \in \mathbb{R}^2$ there is a unique $\gamma \in \mathbb{R}$ such that $(\alpha, 1+r) \in L_\gamma$. In order to indicate that $(\alpha, 1+r)$ belongs to L_γ , we write $L_\gamma(\alpha, 1+r)$. Clearly, $X = \mathbb{R}^2 \setminus \{(1,0)\}$ is a topological space with the subspace topology which is nothing else that the one induced by the canonical topology¹ of $\mathbb{R}^2$.

We can write X as

$$X = \dot{\cup}_{\gamma \in \mathbb{R}} \tilde{L}_\gamma$$

where $\dot{\cup}$ means disjoint union and $\tilde{L}_\gamma$ is the straight line L_γ where the point $(1,0)$ has been dropped.

Define on X the equivalence relation $\sim$ in the following manner:

$$(\alpha', 1+r') \sim (\alpha, 1+r) \Leftrightarrow (\alpha', 1+r') \in L_\gamma(\alpha, 1+r)$$

Define the quotient set $Y = X / \sim$ to be the set of equivalence classes of elements $L_\gamma(\alpha, 1+r)$ of X . The set Y becomes in a topological space if it is equipped with the quotient topology, which is defined of the following way:

$$\tau_Y = \{V \subset Y : V = Q^{-1}(U), U \text{ is an open set in } X\}$$

where $Q : \mathbb{R}^2 \rightarrow Y$ is the canonical map (or quotient map) defined by $(\alpha, 1+r) \rightarrow L_\gamma(\alpha, 1+r)$.

It is well-known that Q is surjective and open map. Moreover, the quotient map Q is characterized

¹ The one generated by the Euclidean norm.

by the following property which is expressed in the following proposition:²

Proposition 3 *If Z is any topological space and $f : Y \rightarrow Z$ is any function, then f is continuous iff $f \circ Q$ is continuous.*

Define $g : Y \rightarrow \mathbb{R}$ to be $g(\tilde{L}_\gamma(\alpha, 1+r)) = \gamma$. This function associates to each line $\tilde{L}_\gamma(\alpha, 1+r)$ the level γ . Notice that f is not defined at $L_1(1, 1+r)$. Define $f : X \rightarrow \mathbb{R}$ as

$$\tilde{f}(\alpha, 1+r) = \begin{cases} \gamma & \text{if } (\alpha, 1+r) \in \tilde{L}_\gamma(\alpha, 1+r) \\ 1 & \text{if } (\alpha, 1+r) \in \tilde{L}_1(\alpha, 1+r) \end{cases}$$

We state the following technical result:

Lemma 6 *The function $\tilde{f} : X \rightarrow \mathbb{R}$ is continuous and equals $g \circ Q$.*

Proof of Lemma 6 First, take any sequence $(\alpha_n, 1+r_n) \neq (1, 1+r)$ converging to $(1, 1+r) \in \tilde{L}_1(1, 1+r)$. The continuity of $\tilde{f}$ follows from the fact that the sequence $f(\alpha_n, 1+r_n)$ tends to $1 = f(1, 1+r)$. In effect, from the definition of $\tilde{f}$ it follows that $\tilde{f}(\alpha_n, 1+r_n) = f(\alpha_n, 1+r_n)$. Using the definition of f we have that

$$f(\alpha_n, 1+r_n) = 1 - \frac{p(\alpha_n - 1)}{1+r_n}$$

Taking the limit as $n \rightarrow \infty$ on both sides of the previous equality and using the fact that $(\alpha_n, 1+r) \rightarrow (1, 1+r)$, we obtain

$$\lim_{n \rightarrow \infty} \tilde{f}(\alpha_n, 1+r_n) = 1 - \frac{p(1-1)}{1+r_n} = 1$$

Thus, the continuity of $\tilde{f}$ follows.

Second, since Q is surjective, we have that for all $\tilde{L}_\gamma(\alpha, 1+r)$ there exists $(\alpha', 1+r') \sim (\alpha, 1+r)$ such that $Q(\alpha', 1+r') = \tilde{L}_\gamma(\alpha, 1+r)$. Notice that $Q(\alpha, 1+r) = Q(\alpha', 1+r')$. So, $Q(\alpha, 1+r) = \tilde{L}_\gamma(\alpha, 1+r)$. Then, from the definition of g , we have $g(Q(\alpha, 1+r)) = g(\tilde{L}_\gamma(\alpha, 1+r)) = \gamma$. But $\gamma = \tilde{f}(\alpha, 1+r)$ as $(\alpha, 1+r) \in \tilde{L}_\gamma(\alpha, 1+r)$. Therefore, we have that $g \circ Q = \tilde{f}$ as $(\alpha, 1+r)$ is arbitrary. This ends the proof.

An immediate consequence of Lemma 6 and Proposition 3 is that g is continuous in the quotient topology.

Let $(\alpha, 1+r)$ satisfy $\tilde{f}(\alpha, 1+r) = 1$. This means that $Q(\alpha, 1+r) = \tilde{L}_1(1, 1+r)$ with $\alpha = 1$. By the definition of g we have that,

$$g\left(\tilde{L}_1(1, 1+r)\right) = 1 \tag{D.4}$$

² See Wyler (1973) for applications of quotient maps.

Equation (D.4) means that the economy is in a complete default state $\gamma = 1$. Consider the sequence $(\alpha + \frac{1}{n}, 1 + r)$. Clearly this sequence converges to $(\alpha, 1 + r)$. Moreover, we have that

$$\tilde{f}\left(\alpha + \frac{1}{n}, 1 + r\right) = f\left(\alpha + \frac{1}{n}, 1 + r\right) = \left(1 - \frac{p(\alpha - 1)}{1 + r}\right) - \frac{p}{n(1 + r)} = 1 - \frac{p}{n(1 + r)}$$

Lemma 2, the previous equality, becomes,

$$(g \circ Q)\left(\alpha + \frac{1}{n}, 1 + r\right) = 1 - \frac{p}{n(1 + r)}$$

which is the same as stating that

$$g\left(\tilde{L}_{\gamma_n}\left(\alpha + \frac{1}{n}, 1 + r\right)\right) = \gamma_n = 1 - \frac{p}{n(1 + r)} \quad (\text{D.5})$$

implying that the economy is a partial-default state, $\gamma_n = 1 - \frac{p}{n(1+r)} < 1$ for n large enough. Continuity of $\tilde{f}$ allows us to show that the sequence $\tilde{L}_{\gamma_n}(p + \frac{1}{n}, 1 + r)$ in Y converges to $L_1(\alpha, 1 + r)$ with respect to the quotient topology τ_Y . Thus, in the limit (D.5) is (D.4) meaning that for any neighborhood small enough of the vertical line $\tilde{L}_1(\alpha, 1 + r)$ representing a completed default $\gamma = 1$, there are infinitely many straight lines³ $L_{\gamma_n}(\alpha + \frac{1}{n}, 1 + r)$ representing a partial default, $\gamma_n \in (1 - p, 1)$ belonging to the earlier neighborhood. This argument is sufficient to demonstrate that the complete default is not robust in the topological sense.

We end this section, remarking on the economic intuition of the two limiting cases above.

Remark 10 *For the case of no default, the intuition behind the argument in Section B.1 is the following: if the economy is at a no-default state and the fiscal shock decreases (from α to $\alpha - \frac{1}{n}$) due to an increase in the leverage for instance, then the government likely makes a partial default on its debts still maintaining the probability of a boom. The intuition for the case of complete default, if the economy is at a complete-default state and the shock fiscal increases (from 1 to $1 + \frac{1}{n}$) due to a decrease in the leverage for instance, then the government likely makes a partial default on its debts.*

³ That is, lines with slope $p(1 - \gamma_n)^{-1}$.

E Sovereign Default Risk Premium under Risk-Neutral Lenders

We consider an economy in which the lender is risk neutral, thus $U(c) = a_0 c_0 + \beta \sum_{s=1}^S p_s a_s c_s$, with $s = 0, 1, \dots, S$, where $a_s > 0$ are the coefficients of the utility function of Lender which is linear. For the sake of simplicity and for comparison purposes, let us consider $S = 2$ and $p > 0$ the probability of a boom. Under Assumption 2 the future primary surplus is $E_1 = E_0 + A_0 > 0$ and $E_0 = E_0 - A_0 < 0$. Given that U is strictly increasing, then the budget constraints hold with equality so that the lender's problem is reduced to

$$\max_{\theta \geq 0} a_0(w_0 - q\theta) + \beta(a_1 p(w_1 + (1 - \pi_1)\theta) + a_2(1 - p)(w_2 + (1 - \pi_2)\theta)). \quad (\text{E.1})$$

The first order condition associated to (E.1) is

$$\beta(a_1 p(1 - \pi_1) + a_2(1 - p)(1 - \pi_2)) \leq a_0 q \text{ with equality if } \theta > 0. \quad (\text{E.2})$$

On the government side we have

$$-E_0 = q\varphi; D_1 = E_0 + A_0, D_2 = 0; 0 \leq D_1 \leq \varphi. \quad (\text{E.3})$$

A default rate rationally anticipated, allied with market clearing imply

$$(1 - \pi_1) = \frac{E_0 + A_0}{\varphi}, 1 - \pi_2 = 0. \quad (\text{E.4})$$

The relations (E.2)–(E.4) characterize the equilibrium of the economy.

E.1 Cost of the Public Debt

In what follows we are going to determine the cost of the public debt both when the sovereign defaults and when it does not. We begin by considering a case particular of Lemma 1 adapted to our case.

Lemma 7 *Under the second relation in (E.4) the following are satisfied*

1. $1 - \tilde{\pi} = (1 - \pi_1)p$
2. $\tilde{\pi} = 0 \Rightarrow (1 - \pi_1)p = 1$
3. $\pi_1 = 0 \Rightarrow 1 - \tilde{\pi} = p$

Proof of Lemma 7 Straightforward.

Next, we determine the cost of the public debt in terms of preferences of the lender. On the one hand, the market clear condition implies $\theta > 0$ as $\varphi > 0$. This is due to the fact that $-E_0 > 0$. On the other hand, it also implies that $q > 0$.

Using (E.2) and Item 1 of Lemma 7 and then solving for $1 + r$ after using Item 1 of Lemma 7 we have

$$1 + r = \frac{a_0}{\beta a_1} \frac{1}{1 - \tilde{\pi}} \quad (\text{E.5})$$

Equation (E.5) represents the cost of the public debt when the sovereign defaults at state $s = 1$. Writing (E.5) in terms of the lender's rate of impatience⁴ $r_1^l = \frac{a_0}{\beta a_1 p} - 1$ at the consumption stream (c_0, c_1) , we have

$$1 + r = \frac{p(r_1^l + 1)}{1 - \tilde{\pi}} \quad (\text{E.6})$$

Making $\pi_1 = 0$ in (E.6) we obtain the the cost of the public debt when there is no default at state $s = 1$. Thus, the risk-free rate of return as defined in Section 7.2 is

$$1 + i = r_1^l + 1 \quad (\text{E.7})$$

as $1 - \tilde{\pi} = p$ by Item 3 of Lemma 7. Lastly, from Item 1 of Lemma 7 it follows that $\frac{p}{1 - \tilde{\pi}} = \frac{1}{1 - \pi_1} > 1$. Thus, combining (E.6) and (E.7) we have

$$1 + i < 1 + r \quad (\text{E.8})$$

Remark 11 If we suppose $a_o > 0$ and $\frac{a_o}{a_1 p} < \beta$, from (E.5) we have $1 - p < \tilde{\pi} < 1$ and therefore there exists default in equilibrium and it is partial.

E.2 Default Risk Premium Under Lender's Characteristics

For the sake of comparison we write (E.2) and we use it to compute the the legitimate risk-free rate of return, namely the rate of return of the economy. The inverse of this rate of return, denoted by $\frac{1}{R}$, is the price of the Treasury debt issued by a sovereign that never defaults ($\pi_1 = 0, \pi_2 = 0$) which is not our case as we dealing with defaulting sovereigns, see Assumption 2. In fact, we have

$$\frac{1}{R} = \frac{\beta a_1}{a_0} p + \frac{\beta a_2}{a_0} (1 - p) \quad (\text{E.9})$$

In terms of the lender's rate of impatience and using (E.7), (E.9) becomes

$$\frac{1}{R} = \frac{1}{1 + i} + \frac{1}{1 + r_2^l} \quad (\text{E.10})$$

⁴ See Magill and Quinzii (1996), p. 128 for a definition.

Clearly from (E.10) and using (E.8) we conclude

$$R < 1 + i < 1 + r$$

Computing the default risk premium as defined in Section 7 we have that

$$r^{pr} = 1 + r - (1 + i) = (r_1^l + 1) \left(\frac{p}{1 - \tilde{\pi}} - 1 \right) = (1 + r_1^l) \frac{\tilde{\pi} - (1 - p)}{1 - \tilde{\pi}} \quad (\text{E.11})$$

Remark 12 *It is useful to point out that if the lender anticipates good times in the sense that the probability of a boom is close to 1, then the premium they require for purchasing bond from defaulting sovereign is almost the same as purchasing Treasury debt. This fact follows from (E.9). In other words, $R \simeq 1 + i$ since $\frac{1}{R} \simeq \frac{\beta a_1}{a_0} p = \frac{1}{1+i}$ as $p \rightarrow 1$.*

E.3 Default Risk Premium Under Fiscal Disturbances

Substituting the first equality in (C.11) into (E.4) we have that

$$(1 - \pi_1) = \frac{\alpha - 1}{1 + r}, 1 - \pi_2 = 0. \quad (\text{E.12})$$

where as before $\alpha = \frac{A_0}{-E_0}$ is the fiscal disturbance.

Making $\pi_1 = 0$ in (E.12) we obtain the the cost of the public debt when there is no default at state $s = 1$. Thus, the risk-free rate of return as defined in Section 7 is

$$1 + i = (\alpha - 1) \quad (\text{E.13})$$

Multiplying by p on both sides of the first relation in (E.12) and solving for $1 + r$ after using Item 1 of Lemma 7 we have

$$1 + r = \frac{p(\alpha - 1)}{1 - \tilde{\pi}} \quad (\text{E.14})$$

Remark 13 *It is useful to note that the cost of the defaulting government's public debt can be determined either in fiscal terms as in (E.14) or in terms of lender's preferences as in (E.5). On the other hand, also notice that both (E.7) and (E.13) give the cost of the public debt when the sovereign does not default. Therefore*

$$1 + r_1^l = \frac{a_0}{\beta a_1 p} = \alpha - 1 \quad (\text{E.15})$$

Using (E.13) and (E.14) we compute the default risk premium:

$$r^{pr} = (1 + r) - (1 + i) = (\alpha - 1) \left(\frac{p}{1 - \tilde{\pi}} - 1 \right) = (\alpha - 1) \frac{\tilde{\pi} - (1 - p)}{1 - \tilde{\pi}} \quad (\text{E.16})$$

Under (E.15) we have that in equilibrium (E.11) and (E.16) are the same.

F Sovereign Default under Positive Primary Surpluses

In Section 4.2 it was implicitly assumed that the future primary surplus did not exceed the claim of the public debt, see inequality (3). However, if, for instance, the government adopts good fiscal policies, the second-period primary surplus could exceed the payoff of the public debt which is being assumed to be risk free. So, given the good fiscal policies, without loss generality we can suppose that $E \in \mathbb{R}_{++}^S$. Nonetheless, there could be states of nature where E_s could be small enough such that it will not cover the debt service. Therefore the public debt would be subject to default. Notice that this kind of default is not strategic since defaulting is not a decision of the government but rather depends on the comparison between the value of future primary surplus and the claim of the public debt. What we can state is that low future primary surpluses depend on fiscal efforts.

Given the good fiscal policies, it is rather likely that there is the set of states of nature, $s \in S$, where $E_s > \varphi$. In that case, the government would end up with a positive balance. Since there are only two periods, that positive balance could be anticipated by issuing another type of security that is not the public debt, namely a derivative whose payoff is $[E_s - \varphi]^+$. If we assume that ϵ is the price of that derivative, then the first-period budget constraint of the government is

$$-E_0 = q\varphi + \epsilon\phi \quad (\text{F.1})$$

and therefore its second-period budget constraint will be

$$\min\{E_s, \varphi\} + [E_s - \phi]^+ = E_s, s \in S. \quad (\text{F.2})$$

Remark 14 Equation (F.1) says that primary surplus $-E_0$ is financed by both the issuance of public debt and sale of the derivative. Equation (F.2) says that the future primary surplus is used to pay the payoffs of both securities issued at the first period. Moreover, for the financial structure (set claims issued) to be feasible (compatible with the future primary surplus), we require $\varphi = \phi$. Finally, if the primary surplus E_s is below φ in all states, then (F.1) and (F.2) reduce to (1), (2) and (3) once $D_s = \min\{E_s, \varphi\} \leq \varphi$.

The lender's problem is to choose consumption $c \in \mathbb{R}_+^{1+S}$, and both public debt and derivative at the same level θ in order to maximize $U : \mathbb{R}_+^{1+S} \rightarrow \mathbb{R}$ subject to the following budget constraints:

$$c_0 + q\theta + \epsilon\psi \leq w_0 \quad (\text{F.3})$$

$$c_s \leq w_s + \min\{E_s^+, \theta\} + [E_s - \psi]^+ \quad (\text{F.4})$$

Note since we are assuming that there is only one creditor, we require $\theta = \psi$. Thus, (F.4) becomes

$$c_s \leq w_s + E_s \quad (\text{F.5})$$

It is also useful to point out that if we define the payment rate as being

$$1 - k_s = \min\{1, \frac{E_s}{\theta}\}, \quad (\text{F.6})$$

then (F.6) and (F.3) implies (5) as $E_s < \varphi$, and likely the value of derivative $\epsilon\psi$ is zero unless the law of one price does not hold (see Cipriani, Fostel and Houser, 2018). In that case, (F.3) reduces to (4).

Definition 5 *An equilibrium for the economy is an array $[q, \epsilon, (c, \theta, \psi), (\varphi, \phi)]$ such that*

1. (c, θ, ψ) maximize $u(\cdot)$ subject to (F.3) and (F.4),
2. φ balances (F.1) and (F.2)
3. Financial markets clear: $\theta = \varphi$, and $\psi = \phi$.

Remark 15 *The equilibrium of Definition 6 is established following the same methodology used in the proof of Theorem 1. However, to guarantee default in equilibrium, more sufficient conditions are needed.*

G On the Partial-Default Cone

When there are only two states of nature the partial-default cone is independent of the probability distribution of states $s = 1, 2$ (as shown in (21')). However when the uncertainty increases reasonably what we have is actually a moving cone. To gain intuition consider $S = 4$. According to Assumption 3 the probability that $s \in \Gamma$ is $p_1 + p_2 < 1$ while the probability $s \in \Gamma'$ is $p_3 + p_4 < 1$. Thus, the probability of a boom is $p_1 + p_2 = p$, and that of a recession, $1 - p$.⁵

From Lemma 1 we have that non default implies that $1 - \gamma = p_1 + p_2 = p$. Therefore

$$1 + i = \left(\frac{p_1 + \frac{p_2}{3}}{p_1 + p_2} \right) \alpha - 1 \quad (\text{G.1})$$

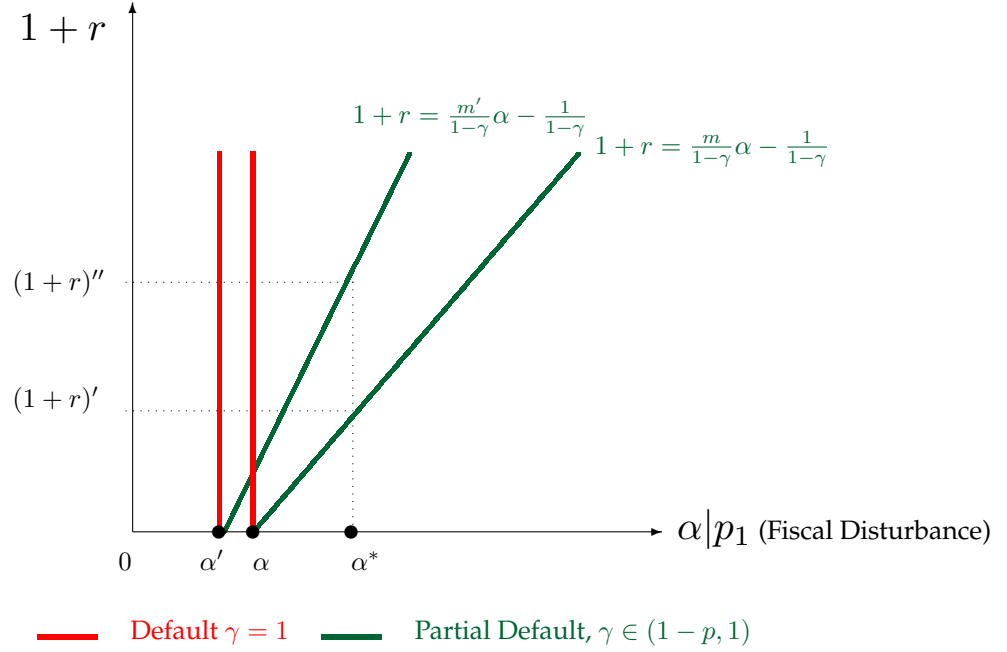
If we denote the slope of the previously straight line by m , then the complete default $\gamma = 1$ is the vertical line $\alpha = \frac{1}{m}$ with $1 + r$ indeterminate. Writing m in terms of p_1 we have $m = \frac{2}{3} \frac{p_1}{p} + \frac{1}{3}$. Thus for $p \in [0, p_1]$ we have $m \in [\frac{1}{3}, 1]$.⁶ Whatever the probability of a boom p is the slope m of the cone always belongs to $[\frac{1}{3}, 1]$ and its vertex point belongs to $[1, 3]$. Independently of the values of $1 + r$, geometrically we will have that the partial-default cone determined by $p_1 \in (0, p)$ will be between the most acute generated by $p_1 = p$ and the more obtuse generated by $p_1 = 0$. We illustrate this in Figure G.1 in the plane $(\alpha, 1 + r)$ with p_1 varying on the interval $[0, p]$ where p is assumed to be the fixed probability of a boom. More precisely, we see two intermediate cones whose slopes of the corresponding oblique sides are $m'(p'_1) > m(p_1)$ with $p'_1 > p_1$ respectively. From the intuitive point of view the geometry of Figure G.1 matching with from second relation in (20) which stated that when p_1 increases the price of debt falls producing a higher gross interest rate. Thus, our analysis says that when p_1 increases the partial-default cone becomes more acute making a steeper slope of line $1 + r$ with a same default level $\gamma \in (1 - p, 1)$ and for a same fiscal disturbance level α . This means that the narrower the cone is, caused by an increase in the probability p_1 , the more the gross interest rate the sovereign must pay keeping both the default level and fiscal disturbances fixed.

Remark 16 A similar analysis can be carried out in the plane $(1 + r, p_1)$ where the narrowness of the partial-default cone depends on fiscal disturbances α . In the same way, we can perform it in the plane $(1 + r, \rho)$ when α is endogenous. The impact of these parameters, α and ρ , on the oblique side of the earlier cone can be drawn from (19) and (36) respectively.

⁵ Recession of course implies states where the future primary surplus is negative: the sovereign does not repay anything.

⁶ This is the same as stating $p_2 \rightarrow p$.

Figure G.1: Partial-Default Cone with Fixed Probabilities of a Boom p



Notes: We have dispensed with the non-default blue line $\gamma = 1 - p$ to make the figure clearer. For $S = 4$, and with $p_1 + p_2 = p$ denoting the given probability of a boom, we have $p_3 + p_4 = 1 - p$ is the probability of recession. Notice that p is kept to be fixed. The value $m = \frac{3}{2} \frac{p_1}{p} + \frac{1}{3}$ is the slope of the non-default line $\gamma = 0$ whose equation is $1 + i = m\alpha - 1$. Thus, the slope of the green line on the right side is $\frac{m}{1-\gamma}$. If we increase p_1 the slope $\frac{m}{1-\gamma}$ increases to the slope $\frac{m'}{1-\gamma}$ of the partial-default line on the left. This implies that the vertex $\alpha = \frac{1}{m}$ moves to the left to reach $\alpha' = \frac{1}{m'}$.

Take a same fixed default level in both cones, say $\gamma \in (1 - p, 1)$. For a fixed fiscal disturbance α^* we have that $(1 + r)'' > (1 + r)'$ due to a increasing of p_1 . This is according to the second inequality in (20).

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