Nonlinear Dynamic Factor Models*

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Abstract

We propose a new class of dynamic factor models featuring nonlinear terms. In simulations, we find that the nonlinear dynamic factor model (NLDFM) performs well with highly volatile series and when the signal-to-noise ratio is low. We show how to use the Unscented Kalman filter and particle filter to estimate the model. We apply our approach to extract an index of economic activity from a set of 7 real and financial variables between 1985 and 2017. In doing so, we respect the nonlinearities imposed by the zero lower bound in the fed funds rate. Our estimation reveals that the nonlinear economic activity index tracks closely the CBO's output gap. Our index differs from the index implied by a linear dynamic factor model in the last decade.

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1 Introduction

We incorporate several innovations to the estimation of economic activity indexes: nonlinearities in measurement and state equations, information about financial conditions, and uncertainty. Nonlinearities have be shown to be important in reduced-form context (TVP, regime switching, and stochastic volatility) and structural models (Fernandez-Villaverde et al. (2015)). Financial and uncertainty information became household names

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following the 2008 crisis. Finally, conventional monetary policy has been constrained until recently as reflected by the effective zero lower bound. Surprisingly, these features have been slowly adopted or not all in the factor model literature. Let alone in conjunction. We propose a factor model, which we call the nonlinear dynamic factor model (NLDM), that respects the zero lower bound in the measurement equation of interest rates, incorporates data on credit spreads and uncertainty, and allows for nonlinear dynamics in the evolution of the underlying factor.

Our nonlinear factor model is inspired by the pruned second/third-order state-space model discussed in Andreasen et al. (2013) as the approximate solution of nonlinear dynamic stochastic general equilibrium models. We re-interpret this framework as one that involves the evolution of a dynamic factor model, whose factor(s) evolves according to the state space model's state equation. We depart from the standard pruned state-space model in that we allow the measurement equation to be potentially nonlinear. This accommodates situations where the observables are bounded and the presence of non-additive measurement errors.

To bring the non-standard features of this state space model to the data, we use the uncented Kalman filter (UKF) and the particle filter (Särkkä (2013)). The former filter has the advantage of simplicity and speed but sacrifices accuracy. The later filter is accurate but slow and suffers from the course of dimensionality. By proposing these two a filtering alternatives, our intention is to provide the practitioner with flexible tools to take our model to the data.

In a simulation exercise, our method estimates a factor that tracks more closely the actual process than the one implied by the Kalman filter. Based on mean squared errors, we find that our approach improves over the Kalman filter as the signal-to-noise ratio declines.

For our empirical application, we apply the nonlinear dynamic factor model to estimate an economic activity index using monthly macro and financial US data since 1985. The estimated index captures business cycles, with positive values almost always coinciding with expansions and negative ones corresponding to times of economic downturns. However, the index tend to stay depressed at the beginning of the recovery face. Compared to other indexes,the recovery stage implied by our model is a bit "delayed." Interestingly, the recover factor tracks closely the Congressional Budget Office's output gap with a correlation of 0.76. We also find that our index moves closely with one implied by a standard linear DFM during the 1990s and early 2000s. However, they part ways around 2003 with our index pointing to a faster rebound from the 2001 recession. Compared to the linear index, the NFM index shows 1) a sharper decline in economic activity during the Great Recession; 2) the recovery starting earlier; and 3) a faster recovery. Indeed, the linear index has the economic below its pre-crisis value even at the end of 2017.

The literature has considered some forms of nonlinearities like time-varying parameters and stochastic volatility. However, to the best of our knowledge, there is no work on models that allow for nonlinear dynamics in the state equation. The classical example these days of a nonlinearity in the data is the zero lower bound imposed on short-term interest rates. In general, we find lower and upper bounds when we deal with percentages like labor-market tightness, transitions probabilities, job finding and separation rates in the labor market.

Related literature: An in depth review of factor models is given by Stock and Watson (2016). Among those recent advances in factor models, we relate to 1) Banbura and Modugno (2014) who allows missing data with arbitrary patterns in the estimation of factor models. Due to missing observations, they use an expectations-maximization algorithm rather than standard techniques; 2) Chauvet (1998) uses a linear factor model but with regime switches to estimate business cycles; 3) Aruoba et al. (2017) introduce the quadratic autoregressive process (QAR), which allows quadratic terms in lagged regressors as well as GARCH features. Like our approach, they rely on the pruned representation to generate a stable model but their approach concentrates on univariate models and no discussion on underlying factors; 4) Aruoba and Diebold (2010) leave nonlinear factors as a to-do task, although with a focus in Markov switching regimes rather than the type we propose; and 5) Cheng et al. (2016) propose a linear DFM that allows for breaks in loadings and/or the number of factors. This is an alternative view of the world. They find that the Great Recession led to a change in the factor loadings and the emergence of a new factor; 6) Carrasco and Rossi (2016) consider forecasting with misspecificed factor models. Like their work, we conduct inference using a misspecified linear factor model. Unlike theirs, the actual DGP features a nonlinear factor in our setup. Finally, we are motivated by the nonlinear representation of dynamic stochastic general equilibrium (Fernandez-Villaverde et al., 2015).

The rest of the paper is organized as follows. The next section discusses the nonlinear dynamic factor model using a simple example with two observables. Using simulations, we address some of the identification issues behind the model. We also show the implementation using the UKF and particle filter. In Section 3, we present our application. Some concluding remarks are in the final section.

2 A Simple NDF Model

Let's consider a simple nonlinear DFM. Let f_t denote the underlying factor and f_t^f and f_t^s the factor's first and second terms: $f_t = f_t^f + f_t^s$. For simplicity, we consider the case of two variables in the measurement equation. Then the pruned system (Andreasen et al. (2013)) is

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \underset{[2\times1]}{H} f_t + \underset{[2\times2]}{\eta} \underset{[2\times1]}{\epsilon_t}, \tag{1}$$

$$f_t^f = h_x f_{t-1}^f + \sigma \nu_t, \tag{2}$$

$$f_t^s = h_x f_{t-1}^s + \frac{1}{2} H_{xx}(f_{t-1}^f \times f_{t-1}^f), \tag{3}$$

$$f_t = f_t^f + f_t^s. \tag{4}$$

The model allows for rich nonlinear dynamics that has not been explored. To see this, let's expand the second-order term of the factor:

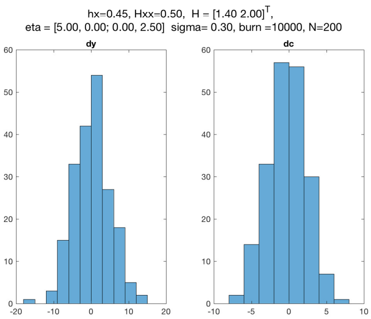
$$f_t^s = h_x f_{t-1}^s + \frac{1}{2} H_{xx} \left(h_x^2 (f_{t-2}^f)^2 + 2 h_x \sigma f_{t-2}^f \nu_{t-1} + \sigma^2 \nu_{t-1}^2 \right).$$

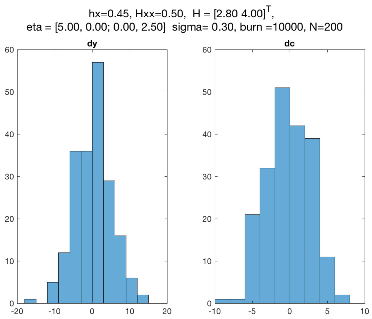
As one can see, the model incorporates time-varying volatility and an interaction between the factor and the factor's innovation. This second element makes the effect of shocks on observables potentially state dependent. This interaction can be interpreted as a restricted time-varying parameter factor model, where the TVP term is i.i.d.

2.1 Simulation

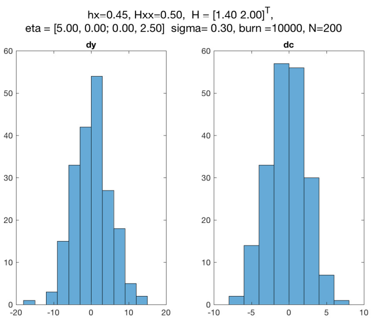
To understand the role of different parameters on the observables, we simulate the NLD model using different parameter configurations. The next figures which display histograms for the observables compare these scenarios.

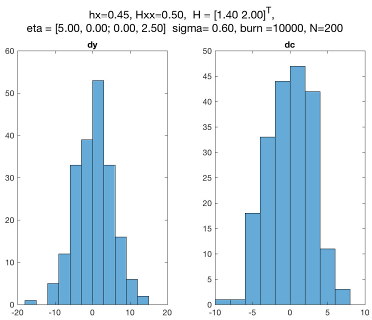
1) Baseline parameters, Double H:



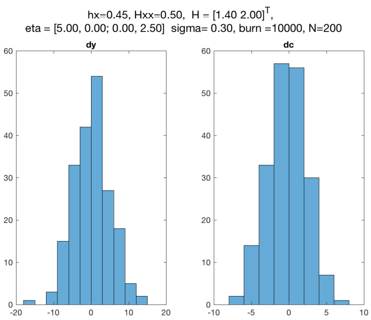


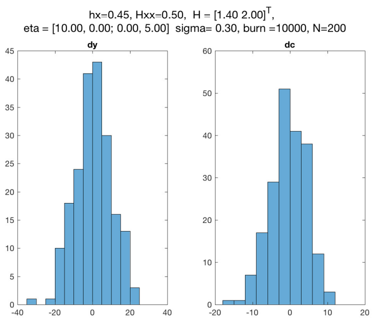
2) Baseline parameters, Double σ



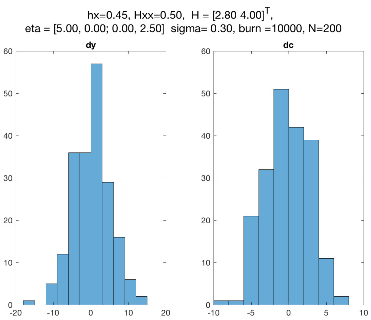


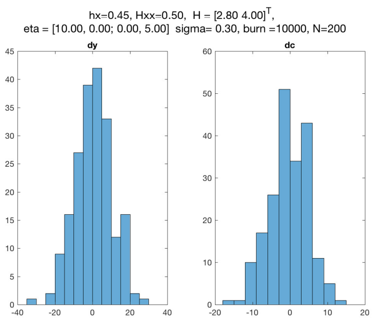
3) Baseline parameters, Double η (each parameter)



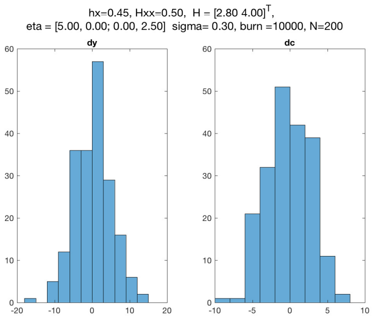


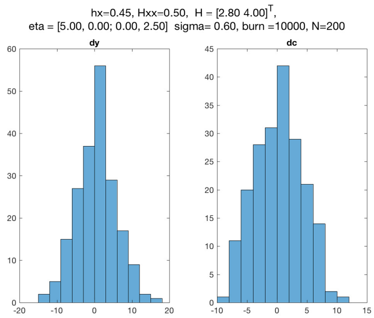
4) Double H, η vs 2η, the rest are baseline.





5) Double H, σ vs 2σ, the rest are baseline.





2.2 Pruned system meets UKF

Recall we are working with a NDF model whose pruned state-space form is:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \underset{2\times 2}{[H, H]} \begin{bmatrix} f_t^f \\ f_t^s \end{bmatrix} + \underset{[2\times 2]}{\eta} \underset{[2\times 1]}{\epsilon_t}, \tag{5}$$

$$f_t^f = h_x f_{t-1}^f + \sigma \nu_t, \tag{6}$$

$$f_t^s = h_x f_{t-1}^s + \frac{1}{2} H_{xx} (f_{t-1}^f \times f_{t-1}^f), \tag{7}$$

The factor's motion equations, which correponds to Sarkka's dynamic model function, is given by the following Matlab function mapping $\mathbf{R}^2 \to \mathbf{R}^2$.

```
function [fvec] = true_state(x, param)
    hx  = param(1);
    Hxx = param(2) ;

    f_f_lag     =   x(1);
    f_s_lag     =   x(2);

    f_f         =   hx*f_f_lag;
    f_s         =   hx*f_s_lag + 0.5*Hxx*f_f_lag^2;

    fvec = [f_f f_s]';
end
```

The presence of the stochastic term in the equation for the first factor is taken into account on the prediction step in the Sarkka's toolbox: matrix Q is given by:

$$Q = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

Zero values reflect the absense of the error term in the second-order equation (7).

Initialization

We initialize the filter at

$$P_0 = \Sigma_X, \quad m_0 = \mu_X,$$

using 10000-period simulation (with 10000 burning period) to estimate the variance-covariance matrix of the factors Σ_X and the mathematical expectation μ_X. [Note: an explicit-form analytical representation is provided in Andreasen et al., 2017]

Filtering

Follows Sarkka's procedure – taking as inputs the measurement function (in our case linear), the dynamic model function, the measurements, and the initial values.

2.3 Performance (vs. K filter in estimation)

Method in brief:

1. For both KF and UKF we chose 50 starting points for the factors (f_0). We run the estimation using each of the starting points . This points are drawn from normal distribution with parameters corresponding to the unconditional distribution of the factors.
2. As initial value for parameters we chose $0.5*$ true values in every case. From the methodological standpoint, we would prefer to choose some initial values not related to the true values. However, in the macroeconomic context choosing values based on some calibration, which would be close to the final estimated values, seems to be a reasonable approach.

Observations:

- For large values of σ, the UKF performs significantly better than the linear state space model estimated using the Kalman filter. measured as mean squared error of the filtered factor f.

- The UKF's advantage diminishes as the signal-to-noise ratio increases.

Table 2.3 shows the estimated values of the model using a misspecified linear model estimated using the Kalman filter (column labeled KF), the estimates using the UKF, and the actual parameters.

	True	KF	UKF
σ^2	3	2.6379	5.0978
$H(1)$	1.4	4.6457	1.9331
$H(2)$	2.0	3.6881	1.8436
Hxx	0.5	-	1.7065
hx	0.45	0.6494	0.2347
$\eta(1,1)$	5	2.9163	5.6460
$\eta(2,2)$	2.5	2.6269	0.0107

Table 1: True and Estimated Parameter Values

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \underset{2\times 2}{[H, H]} \begin{bmatrix} f_{1,t} \\ f_{2,t} \end{bmatrix} + \underset{[2\times 2][2\times 1]}{\eta \quad \epsilon_t},$$

$$f_t^f = h_x f_{t-1}^f + \sigma \nu_t,$$

$$f_t^s = h_x f_{t-1}^s + \frac{1}{2} H_{xx}(f_{t-1}^f \times f_{t-1}^f),$$

Figure 2.3 shows the actual factor (solid line), the Kalman filtered factor (dotted line), and the unscented Kalman filtered factor (dashed line) for a given initial factor f_0. For this simulation, the mean square error for the Kalman filter estimate is 60% larger than the UKF's one. Eyeball econometrics suggests that the gains come from abrupt changes in the data such as those around periods 5, 140, 170, and at the end of the simulation. Overall, the UKF estimate tracks closely the true process.

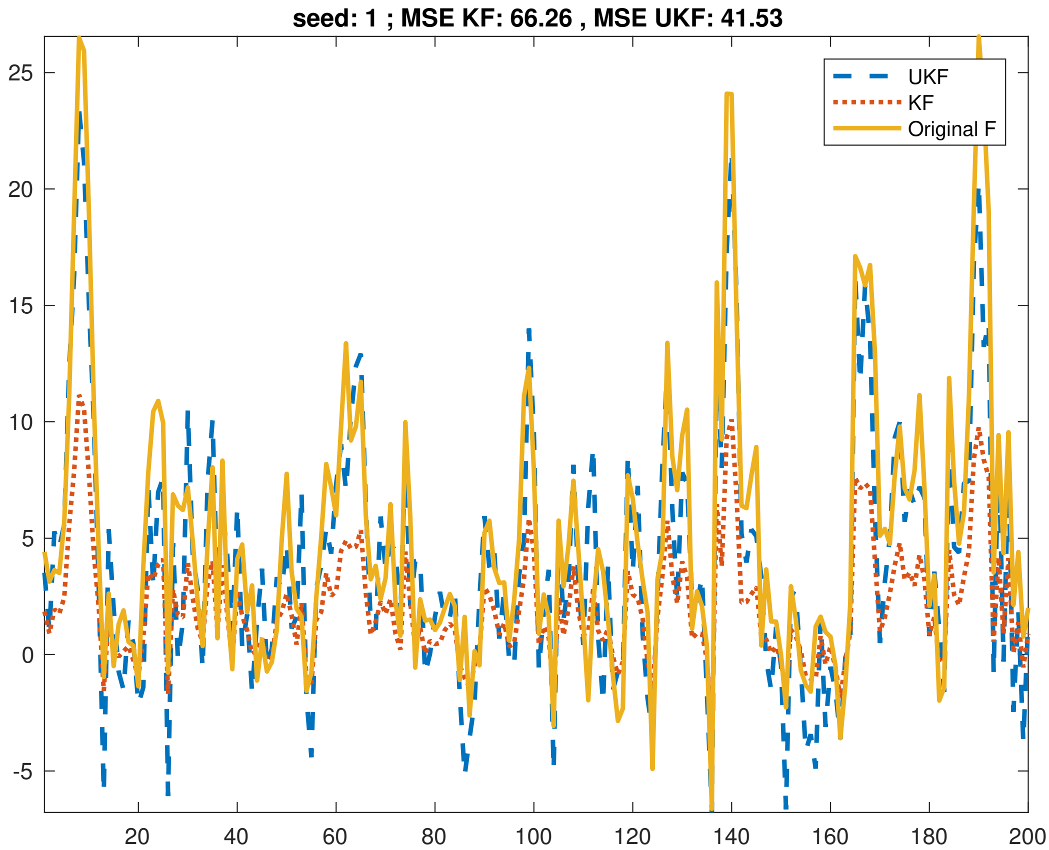


Figure 1: $\sigma = 0.3$; $H = [1.42]'$, $Hxx = 0.3$, $hx = 0.45$, $\eta = [50; 02.5]$

2.4 Extensions: Nonlinear Measurement Equation

Our formulation can accommodate more complex dynamics like nonlinear interactions in the measurement equation. For instance, one could allow for multiplicative measurement errors:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \underset{2\times 2}{[H, H]} \begin{bmatrix} f_{1,t} \times \eta_1 \epsilon_{1,t} \\ f_{2,t} \times \eta_2 \epsilon_{2,t} \end{bmatrix},$$
$$f_t^f = h_x f_{t-1}^f + \sigma \nu_t,$$
$$f_t^s = h_x f_{t-1}^s + \frac{1}{2} H_{xx} (f_{t-1}^f \times f_{t-1}^f),$$

or a fully non-linear measurement equation:

$$
\begin{aligned}
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} &= H\left(\begin{bmatrix} f_{1,t} \times \eta_1 \epsilon_{1,t} \\ f_{2,t} \times \eta_2 \epsilon_{2,t} \end{bmatrix}\right), \\
f_t^f &= h_x f_{t-1}^f + \sigma \nu_t, \\
f_t^s &= h_x f_{t-1}^s + \frac{1}{2} H_{xx}(f_{t-1}^f \times f_{t-1}^f).
\end{aligned}
$$

H is the nonlinear function mapping from measurement errors and factors to observables.

2.5 Extensions: Multidimensional state

One can easily expand our model to accommodate more factors. Below we still use 2 observables but we add an additional factor.

$$
\begin{aligned}
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} &= \underset{2\times 2}{G} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \underset{[2\times 2]}{\eta} \underset{[2\times 1]}{\epsilon_t}, \\
\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} &= \mathcal{H}\left(\begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix}\right) + \underset{2\times 2}{\Sigma} \nu_t.
\end{aligned}
$$

Here, the function $\mathcal{H}$ is the nonlinear map between the factors yesterday and the factors today.

3 Application

As an application of our procedure, we extract an economic activity indicator.

3.1 Data

We use monthly data starting in January 1985 and ending in June 2017. The variables are the fed funds rate, hourly earnings (AHETPI), spread between Baa corporate bond yield and 10-year Treasury (BAA10YM), CPI inflation, industrial production, spread between 10-Year Treasury Constant Maturity and 2-Year Treasury Constant Maturity (T10Y2YM), and weekly hours worked (HOHWMN02USM065S). We take the first log-differences for the non-stationary series: hourly earnings, CPI, and industrial production. The data are retrieved from the St. Louis Fed's FRED database. Figure 2 displays the series. One can see that our choice of observables includes highly volatile series like industrial production and monthly inflation but also more stable series like the spreads. Importantly, the volatile nature of the dataset and the presence of the zero lower bound make our nonlinear DFM suitable to extract an economic activity index. Before the estimation step, we standardize the series to have mean zero and unit standard deviation.

3.2 Model and Estimation Strategy

We use the nonlinear dynamic factor model introduced in the previous section with a modification to respect the zero lower bound in the measurement equation.

$$
\begin{aligned}
\begin{bmatrix} R_t \\ Y_t \end{bmatrix} &= \begin{bmatrix} \Psi(f_t) \\ \underset{[6\times1]}{H} \times f_t \end{bmatrix} + \underset{[7\times7]}{\eta}\underset{[7\times1]}{\epsilon_t}, \\
f_t &= f_t^f + f_t^s, \\
f_t^f &= \phi_1 f_{t-1}^f + \sigma\nu_t, \\
f_t^s &= \phi_1 f_{t-1}^s + \frac{1}{2}\phi_2 (f_{t-1}^f \times f_{t-1}^f),
\end{aligned}
$$

where $\epsilon_t \sim iidN(0, \underset{[6\times1]}{\mathbb{I}})$, $\nu_t \sim iidN(0,1)$. Here, the function $\Psi(\cdot)$ is such that

$$
R_t = \max(h_r f_t, -\frac{\mu_r}{\sigma_r}) + \eta_r \epsilon_t,
$$

where R_t and Y_t are the standardized series of the fed funds rate and the rest of aforementioned series respectively, μ_r and σ_r are the average and the standard deviation of the fed funds series. This allows us to recover a factor that is consistent with the observed interest rate as well as the other variables. We restrict the size of the measurement errors so they don't account for the bulk of the variability observed in the data. For exposition purposes, we let the measurement error be additive, which allows for small deviations from the ZLB. But this is easily fixed by moving the measurement error inside the max operator. The complication is that the UKF may not be directly applied so one may need to resort to the particle filter.

We estimate the parameters of the model using maximum likelihood estimator based on the unscented Kalman filter (UKF) introduced in Julier et al. (1995), Julier and Uhlmann (1996) and in subsequent papers (Julier and Uhlmann (2004), Julier et al. (2000), for details refer to Särkkä (2013)). The technical implementation relies on the UKF from the Matlab toolbox introduced in Hartikainen et al. (2011) under default parameters. For

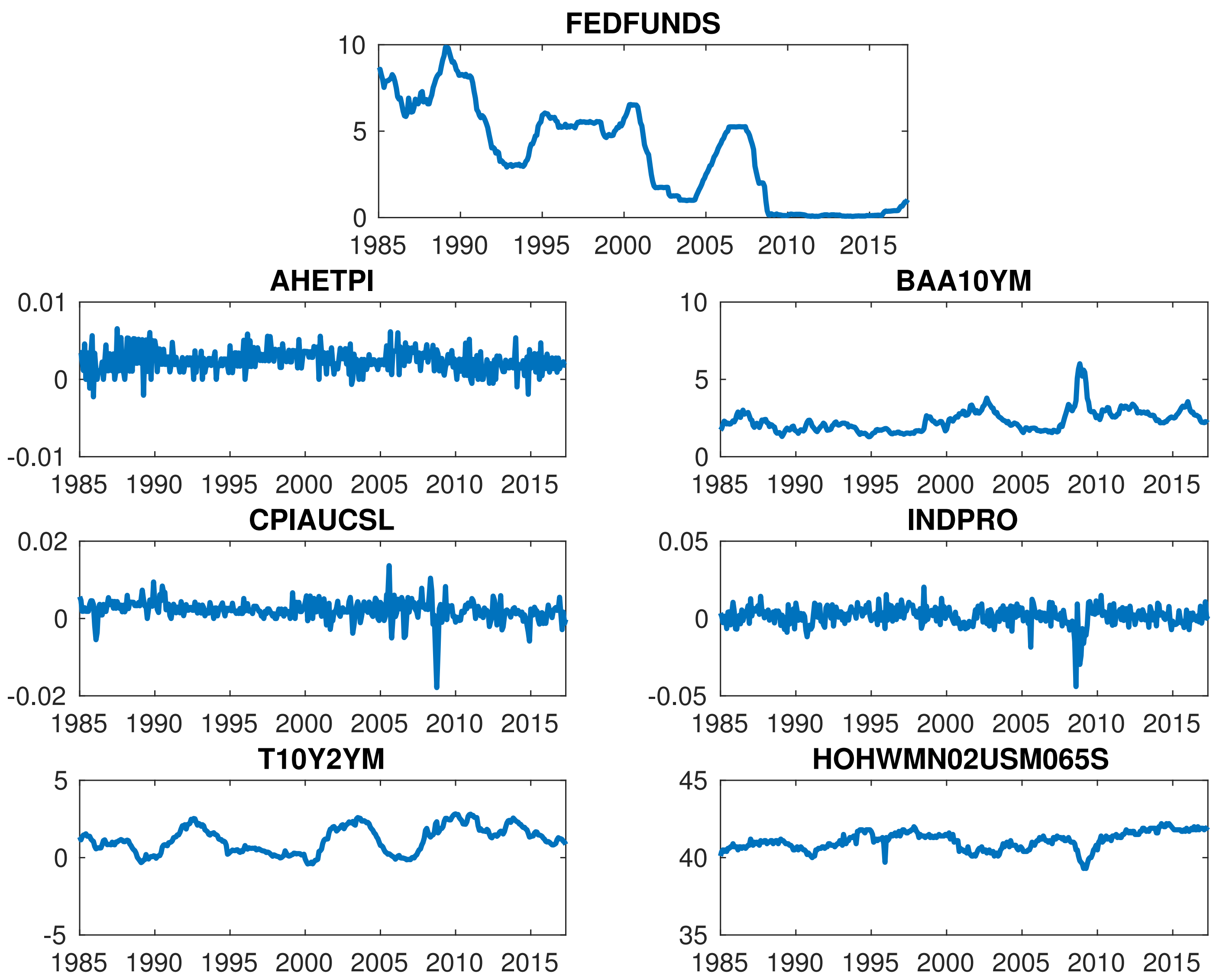


Figure 2: Data used in the estimation.

the maximization we are using Matlab *fminsearch* minimizer tool.[1]

It should be noted, that the estimation of the prunned model is non-trivial from a technical point of view. It would not be correct to say that the model nests the linear factor motion equation, since if $\phi_2 = 0$, there are two state motion equations, only one of which includes the random term, and there is no identification for such model. This leads to a technical problem: if $\hat{\phi}_2 \to 0$ in the process of estimation, the filter would fail[2]. To avoid this, and other associated problems, we add to the regular maximization procedure two elements. First, we set the value of the log-likelihood function to extreme negative value if the filter fails for any reason (divergence when $\phi_1 \geq 1$, the aforementioned singularity of the matrix etc.). Second, to make the maximization procedure work faster we do not allow the maximizing algorithm to go close enough to points at which UKF fails. To implement this, we draw multiple starting points, choose the ones at which UKF fails, and we check at every step of the maximization whether the proposed point is in the proximity of these points.[3]

For simplicity, we set as initial point $[f^f_{1|0}, f^s_{1|0}] = [0, 0]$, and the variance-covariance matrix $Cov_0(f^f_{1|0}, f^s_{1|0}) = 100 \times \underset{[2\times 2]}{\mathbb{I}}$. As the initial values of the parameters we set the point with the maximum value of the likelihood function from the initial draw which we described above. A more refined procedure would include choosing the initial point as a vector of parameters, and the initial variance-covariance matrix – as the result of simulation under the proposed values of parameters.

3.3 Results

The resulting index of economic activity is displayed in Figure 3 (for convenience, we also report recession as shaded areas). Broadly speaking, our index captures business cycles, with positive values almost always coinciding with expansions and negative ones corresponding to times of economic downturns. However, the index tend to stay depressed at the beginning of the recovery face. For example, following the 1991 recession, the index points to a recovery starting two years later. This *delayed recovery* is informed by the dynamics of the fed funds rate and the 10Y-2Y Treasuries spread – it is only in 1993 when rates and spreads start to increase, Figure 2.

[1]It uses simplex search method, for details see the description of the algorithm on Mathworks website and in Lagarias et al. (1998)

[2]The procedure would return $P_{t+1|t} = cov_t(f^f_{t+1|t}, f^s_{t+1|t})$ close to singular, making difficult the Cholesky decomposition of this matrix, which is required for the algorithm.

[3]Introducing too many of such points slows down the process.

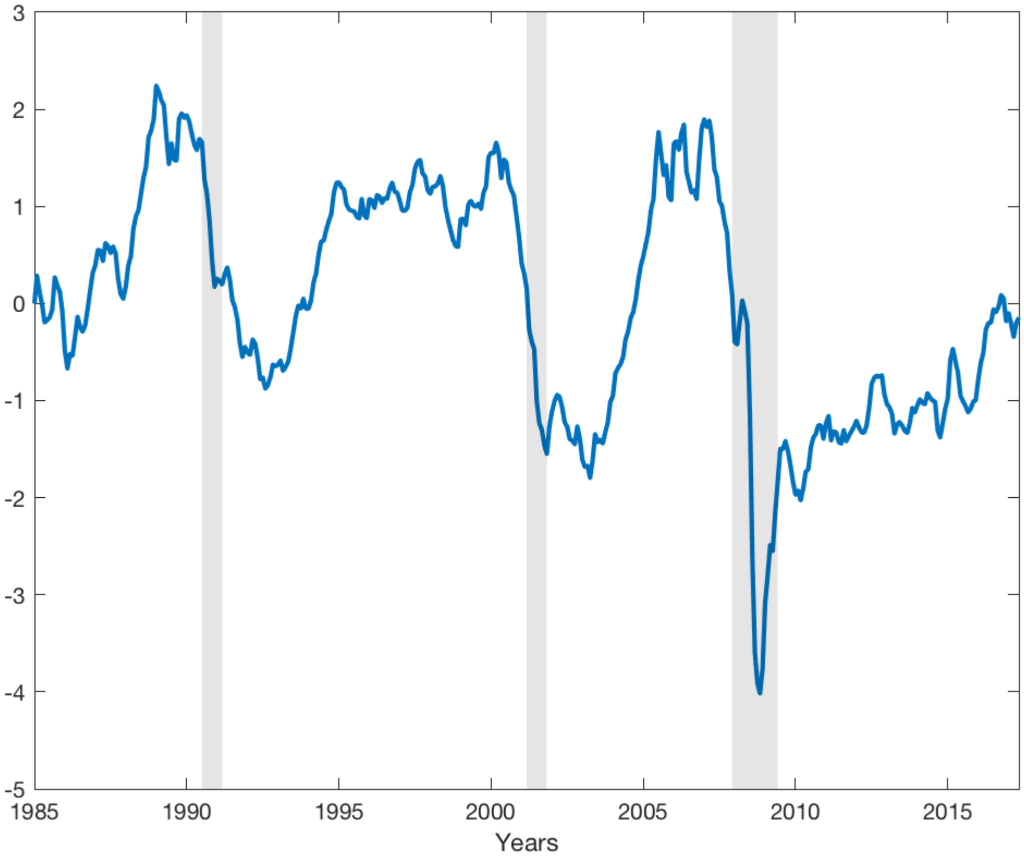


Figure 3: Index of Economic Activity implied by Nonlinear Dynamic Factor Model

The delayed recovery inferred from the index cannot be clearer than in the last recession. In spite of a fast initial rebound, the post-Great-Recession recovery is slow; it is only by 2016 when the index points to the economy getting back to pre-crisis levels. Once again, the fed funds rate seems to inform the dynamics of the pale recovery. Compared to other indexes, ours look a bit "delayed." This raises the question of whether the NFD index has any economic relevance. To shed light on this point, Figure 4 compare our index with the CBO's output gap (red dashed lines). Interestingly, the figure reveals that our index closely follows this definition of the output gap with a correlation of 0.76.

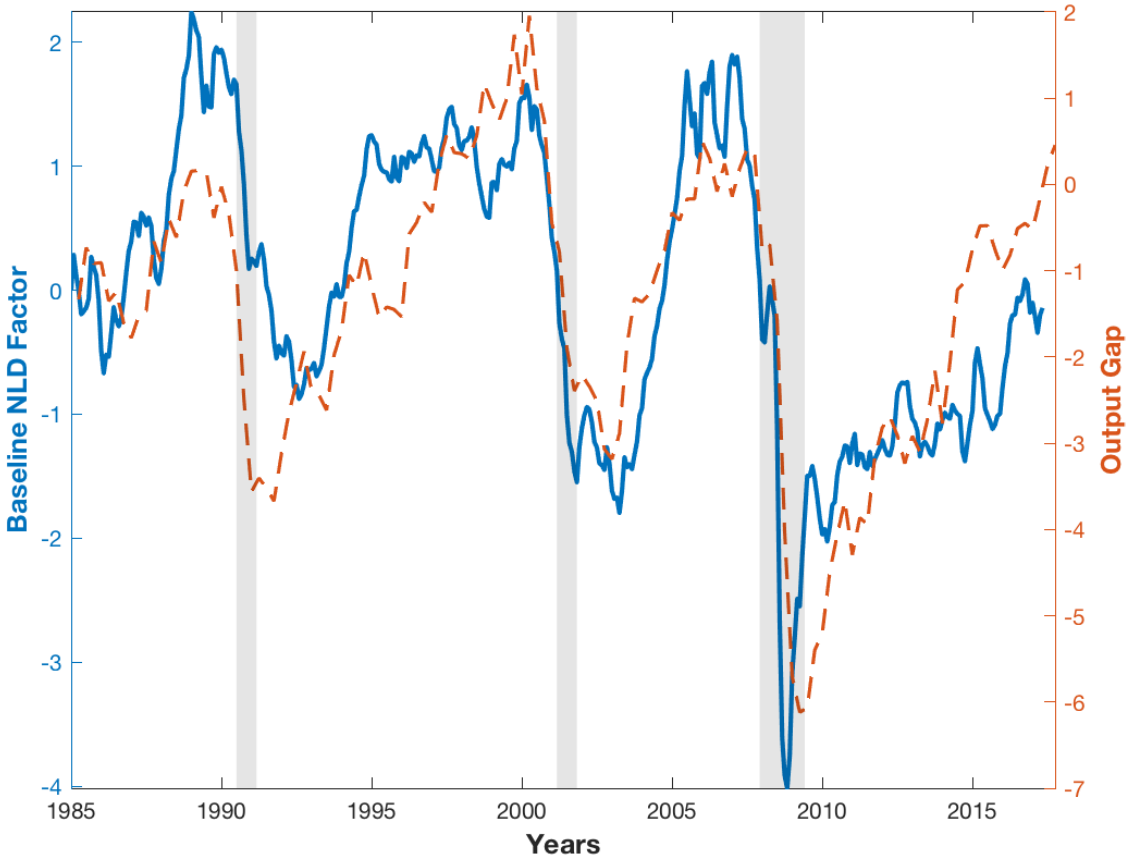


Figure 4: Index of Economic Activity and Output Gap

Figure 5 compares the nonlinear dynamic factor index (blue line) and the one recovered from a standard linear DFM (red dashed line). We can see that our index moves closely with the linear one during the 1990s and early 2000s. However, they part ways around 2003 with our index pointing to a faster rebound from the 2001 recession. Compared to the linear index, the NDFM index shows 1) a sharper decline in economic activity during the Great Recession; 2) the recovery starting earlier; and 3) a faster recovery. Indeed, the linear index has the economic below its pre-crisis value even at the end of 2017.

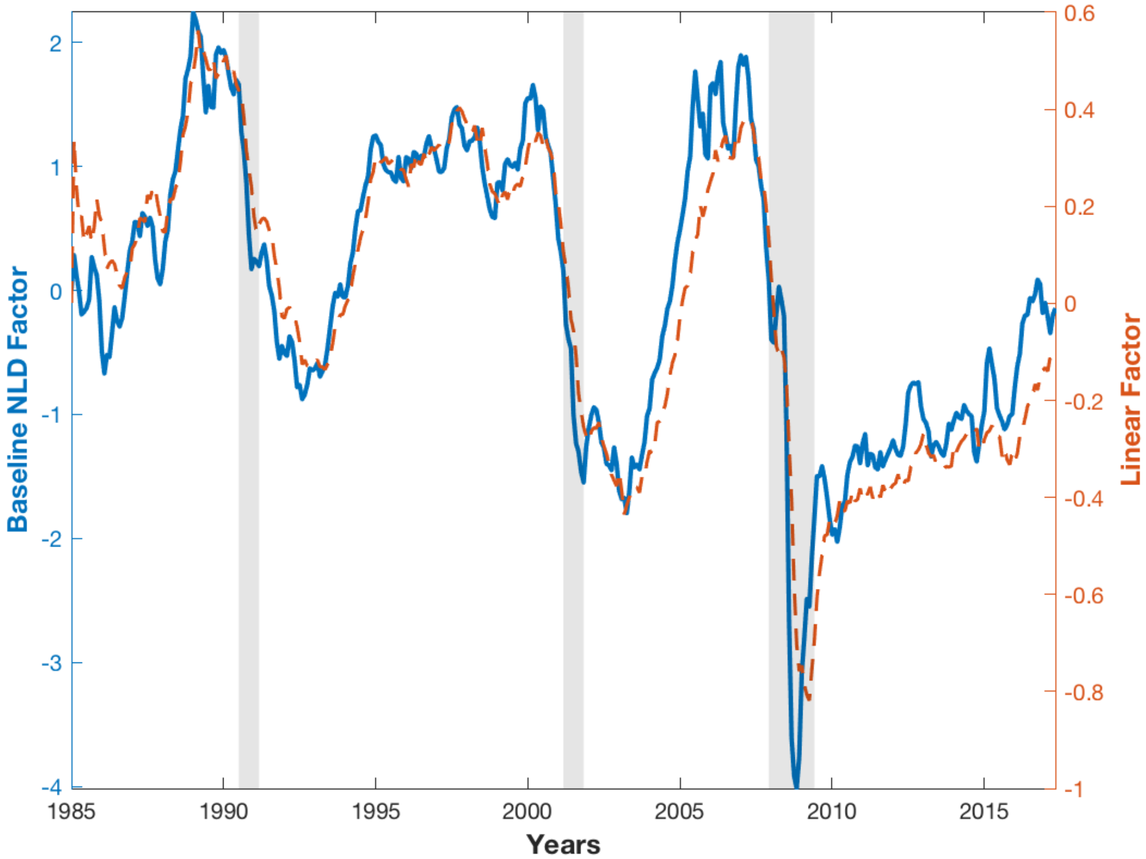


Figure 5: Indexes of Economic Activity implied by Nonlinear Dynamic Factor and Linear Models

4 Conclusion

We propose a parsimonious nonlinear dynamic factor model. Using the Uncented Kalman Filter, we apply this model to estimate an index of economic activity for the U.S., which respects the nonlinearities introduced by the zero lower bound in the interest rates.

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